

# Differential Geometry of Time-Dependent Mechanics

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## Abstract

The usual formulations of time-dependent mechanics start from a given splitting  $Y = \mathbf{R} \times M$  of the coordinate bundle  $Y \rightarrow \mathbf{R}$ . From physical viewpoint, this splitting means that a reference frame has been chosen. Obviously, such a splitting is broken under reference frame transformations and time-dependent canonical transformations. Our goal is to formulate time-dependent mechanics in gauge-invariant form, i.e., independently of any reference frame. The main ingredient in this formulation is a connection on the bundle  $Y \rightarrow \mathbf{R}$  which describes an arbitrary reference frame. We emphasize the following peculiarities of this approach to time-dependent mechanics. A phase space does not admit any canonical contact or presymplectic structure which would be preserved under reference frame transformations, whereas the canonical Poisson structure is degenerate. A Hamiltonian fails to be a function on a phase space. In particular, it can not participate in a Poisson bracket so that the evolution equation is not reduced to the Poisson bracket. This fact becomes relevant to the quantization procedure. Hamiltonian and Lagrangian formulations of time-dependent mechanics are not equivalent. A degenerate Lagrangian admits a set of associated Hamiltonians, none of which describes the whole mechanical system given by this Lagrangian.

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# 1 Introduction

There is an extensive literature both on autonomous [1, 2, 24, 35] and time-dependent mechanics [6, 10, 14, 33, 39, 40] (this list of references is of course far from being exhaustive). The mechanics of autonomous systems is phrased in terms of symplectic geometries on even-dimensional manifolds, in particular, on the cotangent bundle  $T^*M$  of a manifold  $M$ . At the same time, the usual formulations of time-dependent mechanics are developed on  $\mathbf{R} \times T^*M$ . From physical viewpoint, this means that some reference frame has been chosen.

In this paper, our goal is the formulation of time-dependent mechanics in gauge-invariant form, i.e., independently of any reference frame. The main ingredient in this formulation is a bundle  $Y \rightarrow X$  over a 1-dimensional base  $X$ . In such a context, a complete connection on  $Y \rightarrow X$  defines a reference frame. From the mathematical viewpoint, a complete connection on  $Y \rightarrow X$  is equivalent to give a splitting  $Y \simeq X \times M$ . This, in turn, implies the splitting of the covariant phase space  $\Pi = V^*Y \simeq X \times T^*M$  (where  $V^*Y$  denotes the vertical cotangent bundle of  $Y \rightarrow X$ ).

We emphasize the following peculiarities of time-dependent mechanics.

(i) The phase space does not admit any canonical contact or presymplectic structure which would be maintained under changes of reference frames. At the same time, we have the canonical Poisson structure, but it is necessarily degenerate.

(ii) The Hamiltonian fails to be a function on a phase space (see (1.2)). In particular, it can not participate in a Poisson bracket. It follows that the evolution equation is not reduced to the Poisson bracket. As a consequence, integrals of motion can not be defined as functions in involution with the Hamiltonian.

(iii) Canonical transformations fail to admit even local generating functions in general.

(iv) The spray evolution equation is not maintained under general reference frame transformations.

(v) Hamiltonian and Lagrangian formulations of time-dependent mechanics are not equivalent. A degenerate Lagrangian admits a set of associated Hamiltonians none of which describes the whole mechanical system given by this Lagrangian. At the same time, we have not any canonical Poisson or contact structure on a configuration space of Lagrangian mechanics.

We develop time-dependent mechanics as the particular case of field theory on bundles  $Y \rightarrow X$  over a  $n$ -dimensional base [6, 12, 48]. In this approach, the physical variables are described by sections of  $Y \rightarrow X$  (where  $\dim X > 1$  in field theory and  $\dim X = 1$  in mechanics). If  $n > 1$ , the Hamiltonian partner of the first order Lagrangian machinery is the polysymplectic Hamiltonian formalism [11, 18, 25, 27, 46, 48]. If  $n = 1$ , we show that this formalism provides the differential geometric description of time-dependent Hamiltonian

mechanics. In particular, the  $n = 1$  polysymplectic Hamiltonian form is exactly the integral invariant of Poincaré–Cartan

$$H = p_i dy^i - \mathcal{H} dt \tag{1.1}$$

[2, 35] on the phase space  $V^*Y$  coordinatized by  $(t, y^i, p_i)$ . The form (1.1) is written in gauge-invariant form where the Hamiltonian  $\mathcal{H}$  is not a function, but a section of the affine bundle  $T^*Y \rightarrow V^*Y$ . The bundle  $T^*Y \rightarrow X$  coordinatized by  $(t, y^i, p, p_i)$  plays the role of the phase space in the homogeneous formalism. We have the splitting of a Hamiltonian

$$\mathcal{H} = p_i \Gamma^i + \widetilde{\mathcal{H}}, \tag{1.2}$$

where  $\widetilde{\mathcal{H}}$  is a Hamiltonian function and  $\Gamma$  is a connection on  $Y \rightarrow X$  describing a (local) reference frame.

By analogy with field theory, we may talk of *sui generis* gauge-invariant formulation of mechanics. Such formulation enables us both to describe a mechanical system without a preferable reference frame (e.g., in relativistic mechanics and gravitation theory) and to analyze phenomena which depend essentially on the choice of a reference frame (e.g., the energy-momentum conservation laws). Also quantizations with respect to different reference frames are not equivalent.

## 2 Preliminaries

This Section includes the main notions of differential geometry and jet formalism which we need in sequel.

From a pragmatic viewpoint, we widely use coordinate expressions, but all objects satisfy the corresponding transformation laws and are globally defined.

Throughout, morphisms are smooth mappings of class  $C^\infty$ . Manifolds are real, finite-dimensional, paracompact and connected.

**Remark 2.1.** The only 1-dimensional manifolds obeying these conditions are the real line  $\mathbf{R}$  and the circle  $S^1$ . ●

We use the symbols  $\otimes$ ,  $\vee$  and  $\wedge$  for tensor, symmetric and exterior products respectively. By  $\rfloor$  is meant the contraction of multivectors and differential forms. The natural projections of the product  $A \times B$  are denoted by

$$\text{pr}_1 : A \times B \rightarrow A, \quad \text{pr}_2 : A \times B \rightarrow B.$$

Let  $Z$  be a manifold coordinatized by  $(z^\lambda)$ . The tangent bundle  $TZ$  and the cotangent bundle  $T^*Z$  of  $Z$  are equipped with the induced coordinates  $(z^\lambda, \dot{z}^\lambda)$  and  $(z^\lambda, \dot{z}_\lambda)$  relative to the holonomic fibre bases  $\{\partial_\lambda\}$  and  $\{dz^\lambda\}$  for  $TZ$  and  $T^*Z$  respectively. By  $Tf : TZ \rightarrow TZ'$  is meant the tangent morphism to morphism  $f : Z \rightarrow Z'$ .

We recall here the following kinds of morphisms: immersion, imbedding, submersion, and projection. A morphism  $f : Z \rightarrow Z'$  is called *immersion* if the tangent morphism  $Tf$  at every point  $z \in Z$  is an injection. When  $f$  is both an immersion and an injection, its image is said to be a submanifold of  $Z'$ . A submanifold which also is a topological subspace is called imbedded submanifold. A mapping  $f : Z \rightarrow Z'$  is called *submersion* if the tangent morphism  $Tf$  at every point  $z \in Z$  is a surjection. If  $f$  is both a submersion and a surjection, it is termed *projection* or *fibred manifold*.

### 2.1 Bundles

Let  $\pi : Y \rightarrow X$  be a fibred manifold over the base  $X$ . It is provided with an atlas of fibred coordinates  $(x^\lambda, y^i)$ , where  $(x^\lambda)$  are coordinates of  $X$ .

Hereafter, by a *bundle* is meant a locally trivial fibred manifold, i.e. there exists an open covering  $\{U_\xi\}$  of  $X$  and local diffeomorphisms

$$\psi_\xi : \pi^{-1}(U_\xi) \rightarrow U_\xi \times V,$$

where  $V$  is the standard fibre of  $Y$ . The collection

$$\Psi = \{U_\xi, \psi_\xi, \rho_{\xi\zeta}\}, \quad \psi_\xi(y) = (\rho_{\xi\zeta} \circ \psi_\zeta)(y), \quad y \in \pi^{-1}(U_\xi \cap U_\zeta),$$

of the splittings  $\psi_\xi$  together with the transition functions  $\rho_{\xi\zeta}$  constitute a *bundle atlas* of  $Y$ . The associated bundle coordinates of  $Y$  are

$$y^i(y) = (v^i \circ \text{pr}_2 \circ \psi_\xi)(y), \quad \pi(y) \in U_\xi,$$

where  $(v^i)$  are fixed coordinates of the standard fibre  $V$  of  $Y$ . A bundle  $Y \rightarrow X$  is said to be *trivializable* if it admits a global splitting  $Y \simeq X \times V$ . Different such splittings differ from each other in projections of  $Y$  onto  $V$ .

**THEOREM 2.1.** Every bundle over a contractible paracompact manifold is trivializable.  $\square$

**THEOREM 2.2.** If the standard fibre of a bundle over a paracompact base is diffeomorphic to  $\mathbf{R}^m$ , this bundle has a global section.  $\square$

By a *bundle morphism* of  $Y \rightarrow X$  to  $Y' \rightarrow X'$  is meant a pair of mappings  $(\Phi, f)$  which form the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

One says that  $\Phi$  is a bundle morphism over  $f$  (or over  $X$  if  $f = \text{Id}_X$ ).

Given a bundle  $Y \rightarrow X$ , every mapping  $f : X' \rightarrow X$  yields a bundle  $f^*Y$  over  $X'$  which is called *pullback* of the bundle  $Y$  by  $f$ . The fibre of  $f^*Y$  at a point  $x' \in X'$  is that of  $Y$  at the point  $f(x') \in X$ . In particular, the product of bundles  $\pi : Y \rightarrow X$  and  $\pi' : Y' \rightarrow X$  over  $X$  is the pullback

$$\pi^*Y' = \pi'^*Y = Y \times_X Y'.$$

For the sake of simplicity, we shall denote the pullbacks  $Y \times_X TX$ ,  $Y \times_X T^*X$  of tangent and cotangent bundles of  $X$  by  $TX$  and  $T^*X$ .

**Remark 2.2.** Let  $\pi : Y \rightarrow X$  be a bundle. Every diffeomorphism  $\rho$  of a manifold  $Y$  which does not preserve the fibration  $\pi$  defines a new bundle  $\pi \circ \rho^{-1} : Y \rightarrow X$ . Obviously,  $\rho$  is isomorphism of the bundle  $\pi$  to the bundle  $\pi \circ \rho^{-1}$  over  $X$ . At the same time, fibrations  $\pi$  and  $\pi \circ \rho^{-1}$  of  $Y$  are not equivalent. Let  $\rho$  be a bundle isomorphism of  $Y \rightarrow X$  over  $X$ . Given an atlas  $\Psi = \{U_\xi, \psi_\xi\}$  of  $Y$ , there always exists the atlas

$$\Psi \circ \rho^{-1} = \{U_\xi, \psi'_\xi = \psi_\xi \circ \rho^{-1}\} \quad (2.1)$$

of  $Y$  such that the bundle coordinates of points  $\rho(y)$  with respect to  $\Psi \circ \rho^{-1}$  coincide with the bundle coordinates of points  $y$  with respect to  $\Psi$ .  $\bullet$

## 2.2 Differential forms and multivectors

In this Section  $Z$  is an  $m$ -dimensional manifold.

An exterior differential  $r$ -form (or simply a  $r$ -form)  $\phi$  on a manifold  $Z$  is a section of the bundle  $\overset{r}{\wedge} T^*Z \rightarrow Z$ . The 1-forms are called the *Pfaffian forms*. We utilize the coordinate expression

$$\phi = \phi_{\lambda_1 \dots \lambda_r} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r}, \quad |\phi| = r,$$

where the summation is over all ordered collections  $(\lambda_1, \dots, \lambda_r)$ . We denote by  $\mathcal{O}^r(Z)$  and  $\mathcal{O}(Z)$  the vector space of  $r$ -forms and the  $\mathbf{Z}$ -graded algebra of all differential forms on a manifold  $Z$  respectively.

Given a map  $f : Z \rightarrow Z'$ , by  $f^*\phi$  is meant the pullback on  $Z$  of a form  $\phi$  on  $Z'$  by  $f$ . We recall the relations

$$f^*(\phi \wedge \sigma) = f^*\phi \wedge f^*\sigma, \quad df^*\phi = f^*(d\phi).$$

*Contraction* of a vector field  $u = u^\mu \partial_\mu$  and a  $r$ -form  $\phi$  on  $Z$  is given in coordinates by

$$u \rfloor \phi = u^\mu \phi_{\mu \lambda_1 \dots \lambda_{r-1}} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_{r-1}}.$$

There is the relation

$$u \rfloor (\phi \wedge \sigma) = u \rfloor \phi \wedge \sigma + (-1)^{|\phi|} \phi \wedge u \rfloor \sigma.$$

The *Lie derivative*  $\mathbf{L}_u \phi$  of an exterior form  $\phi$  along a vector field  $u$  is defined to be

$$\mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi).$$

It satisfies the relation

$$\mathbf{L}_u(\phi \wedge \sigma) = \mathbf{L}_u \phi \wedge \sigma + \phi \wedge \mathbf{L}_u \sigma.$$

**Example 2.3.** Let  $\Omega$  be a 2-form on  $Z$ . It defines the bundle morphism

$$\boxed{\Omega^\flat : TZ \rightarrow T^*Z, \quad \Omega^\flat(v) \stackrel{\text{def}}{=} -v \rfloor \Omega(z), \quad v \in T_z Z.} \quad (2.2)$$

In coordinates, if  $\Omega = \Omega_{\mu\nu} dz^\mu \wedge dz^\nu$  and  $v = v^\mu \partial_\mu$ , then

$$\Omega^\flat(v) = -\Omega_{\mu\nu} v^\mu dz^\nu.$$

One says that  $\Omega$  is of constant rank  $k$  if the corresponding morphism (2.2) is of constant rank  $k$  (i.e.,  $k$  is the greatest integer  $n$  such that  $\Omega^n$  is not the zero form). The rank of



a *nondegenerate* 2-form is equal to  $\dim Z$ . A nondegenerate closed 2-form is called the *symplectic form*. ●

A *multivector field*  $\vartheta$  of degree  $r$  (or simply a  $r$ -vector field) on a manifold  $Z$  is a section of the bundle  $\overset{r}{\wedge} TZ \rightarrow Z$ . It is given by the coordinate expression

$$\vartheta = \vartheta^{\lambda_1 \dots \lambda_r} \partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_r}, \quad |\vartheta| = r,$$

where summation is over all ordered collections  $(\lambda_1, \dots, \lambda_r)$ .

We denote by  $\mathcal{V}^r(Z)$  and  $\mathcal{V}(Z)$  the vector space of  $r$ -vector fields and the  $\mathbf{Z}$ -graded algebra of all multivector fields on a manifold  $Z$  respectively. The latter is provided with the *Schouten-Nijenhuis bracket*

$$[\cdot, \cdot] : \mathcal{V}^r(Z) \times \mathcal{V}^s(Z) \rightarrow \mathcal{V}^{r+s-1}(Z)$$

which generalizes the Lie bracket of vector fields [5, 52]. This bracket has the coordinate expression

$$\begin{aligned} \vartheta &= \vartheta^{\lambda_1 \dots \lambda_r} \partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_r}, & v &= v^{\alpha_1 \dots \alpha_s} \partial_{\alpha_1} \wedge \dots \wedge \partial_{\alpha_s}, \\ [\vartheta, v] &= \vartheta \star v + (-1)^{|\vartheta||v|} v \star \vartheta, \\ \vartheta \star v &= \vartheta^{\mu \lambda_1 \dots \lambda_{r-1}} \partial_{\mu} v^{\alpha_1 \dots \alpha_s} \partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_{r-1}} \wedge \partial_{\alpha_1} \wedge \dots \wedge \partial_{\alpha_s}. \end{aligned}$$

There are the relations

$$\begin{aligned} [\vartheta, v] &= (-1)^{|\vartheta||v|} [v, \vartheta], \\ [\nu, \vartheta \wedge v] &= [\nu, \vartheta] \wedge v + (-1)^{|\nu||\vartheta|+|\vartheta|} \vartheta \wedge [\nu, v], \\ (-1)^{|\nu||\vartheta|+|\nu|} [\nu, \vartheta \wedge v] &+ (-1)^{|\vartheta||\nu|+|\vartheta|} [\vartheta, v \wedge \nu] + (-1)^{|\nu||\vartheta|+|\nu|} [v, \nu \wedge \vartheta] = 0. \end{aligned}$$

**Example 2.4.** Let  $w = w^{\mu\nu} \partial_{\mu} \wedge \partial_{\nu}$  be a *bivector field*. We have

$$[w, w] = w^{\mu\lambda_1} \partial_{\mu} w^{\lambda_2\lambda_3} \partial_{\lambda_1} \wedge \partial_{\lambda_2} \wedge \partial_{\lambda_3}. \quad (2.3)$$

Every bivector field  $w$  on a manifold  $Z$  yields the associated bundle morphism  $w^{\sharp} : T^*Z \rightarrow TZ$  defined by

$$\boxed{w^{\sharp}(p) \rfloor q \stackrel{\text{def}}{=} w(z)(p, q), \quad w^{\sharp}(p) = w^{\mu\nu}(z) p_{\mu} \partial_{\nu}, \quad p, q \in T_z^*Z.} \quad (2.4)$$

A bivector field  $w$  whose bracket (2.3) vanishes is called the *Poisson bivector field*. ●

Elements of the tensor product  $\mathcal{O}^r(Z) \otimes \mathcal{V}^1(Z)$  are called the *tangent-valued  $r$ -forms* on  $Z$ . They are sections

$$\phi = \phi_{\lambda_1 \dots \lambda_r}^{\mu} dz^{\lambda_1} \wedge \dots \wedge dz^{\lambda_r} \otimes \partial_{\mu}$$

of the bundle  $\overset{r}{\wedge} T^*Z \otimes TZ \rightarrow Z$ . Tangent-valued 1-forms are usually termed the *(1,1) tensor fields*.

**Example 2.5.** There is the 1:1 correspondence between the tangent-valued 1-forms on  $Z$  and the linear bundle morphisms

$$\phi : TZ \rightarrow TZ, \quad \phi : T_z Z \ni v \mapsto v \rfloor \phi(z) \in T_z Z \quad (2.5)$$

over  $Z$ . In particular, the *canonical tangent-valued 1-form*  $\theta_Z = dz^\lambda \otimes \partial_\lambda$  defines the identity morphism of  $TZ$ . •

### 2.3 Tangent and cotangent bundles of bundles

The tangent bundle  $TY \rightarrow Y$  of a bundle  $Y \rightarrow X$  has the *vertical tangent subbundle*  $VY$  given by the coordinate condition  $\dot{x}^\lambda = 0$ . It is coordinatized by  $(x^\lambda, y^i, \dot{y}^i)$  with respect to the holonomic fibre bases  $\{\partial_i\}$ . Given a bundle morphism  $\Phi : Y \rightarrow Y'$ , the restriction of the tangent morphism  $T\Phi$  to  $VY \subset TY$  is the *vertical tangent morphism*

$$V\Phi : VY \rightarrow VY', \quad \dot{y}'^i \circ V\Phi = \dot{y}^j \partial_j \Phi^i.$$

We shall utilize the notation

$$\partial_V = \dot{y}^j \frac{\partial}{\partial y^j}. \quad (2.6)$$

**Example 2.6.** If  $Y \rightarrow X$  and an affine bundle modelled on the vector bundle  $\overline{Y} \rightarrow X$ , there are the canonical isomorphisms

$$V\overline{Y} = \overline{Y} \times_X \overline{Y}, \quad VY = Y \times_X \overline{Y}.$$

•

The *vertical cotangent bundle*  $V^*Y \rightarrow Y$  of  $Y \rightarrow X$  is defined to be the vector bundle dual to the vertical tangent bundle  $VY \rightarrow Y$ . It is not a subbundle of the cotangent bundle  $T^*Y$ . We shall denote by  $\{\overline{dy}^i\}$  the fibre bases for  $V^*Y$  which are dual to the fibre bases  $\{\partial_i\}$  for  $VY$ .

We have the following exact sequences:

$$0 \rightarrow VY \hookrightarrow TY \xrightarrow{Y} Y \times_X TX \rightarrow 0, \quad (2.7a)$$

$$0 \rightarrow Y \times_X T^*X \hookrightarrow T^*Y \xrightarrow{Y} V^*Y \rightarrow 0. \quad (2.7b)$$

Any splitting

$$\Gamma : TX \underset{Y}{\hookrightarrow} TY, \quad \partial_\lambda \mapsto \partial_\lambda + \Gamma_\lambda^i(y)\partial_i, \quad (2.8a)$$

$$\Gamma : V^*Y \underset{Y}{\hookrightarrow} T^*Y, \quad \bar{d}y^i \mapsto dy^i - \Gamma_\lambda^i(y)dx^\lambda, \quad (2.8b)$$

of these sequences corresponds to the choice of a connection on the bundle  $Y \rightarrow X$ .

Let us consider the bundles  $TT^*X$  and  $T^*TX$ . Given the coordinates  $(x^\lambda, p_\lambda = \dot{x}_\lambda)$  of  $T^*X$  and  $(x^\lambda, v^\lambda = \dot{x}^\lambda)$  of  $TX$ , these bundles are coordinatized by  $(x^\lambda, p_\lambda, \dot{x}^\lambda, \dot{p}_\lambda)$  and  $(x^\lambda, v^\lambda, \dot{x}_\lambda, \dot{v}_\lambda)$  respectively. By inspection of the coordinate transformation laws, one can show that they are isomorphic over  $TX$  (see also [13, 27]):

$$\boxed{TT^*X \underset{TX}{=} T^*TX, \quad p_\lambda \longleftrightarrow \dot{v}_\lambda, \quad \dot{p}_\lambda \longleftrightarrow \dot{x}_\lambda.}$$

Given a bundle  $Y \rightarrow X$ , the similar isomorphism of the bundles  $VV^*Y$  and  $V^*VY$  over  $VY$  takes place. In coordinates  $(x^\lambda, y^i, p_i = \dot{y}_i)$  of  $V^*Y$  and  $(x^\lambda, y^i, v^i = \dot{y}^i)$  of  $VY$ , this isomorphism reads

$$\boxed{VV^*Y \underset{VY}{=} V^*VY, \quad p_i \longleftrightarrow \dot{v}_i, \quad \dot{p}_i \longleftrightarrow \dot{y}_i.} \quad (2.9)$$

## 2.4 Forms and vector fields on bundles

Let  $\pi : Y \rightarrow X$  be a bundle coordinatized by  $(x^\lambda, y^i)$ .

A vector field  $u$  on  $Y \rightarrow X$  is termed *projectable* when there is a vector field  $\tau$  on  $X$  such that

$$T\pi \circ u = \tau \circ \pi.$$

Its coordinate expression reads

$$u = u^\mu(x)\partial_\mu + u^i(y)\partial_i, \quad \tau = u^\mu(x)\partial_\mu.$$

A *vertical* vector field  $u = u^i\partial_i$  is a projectable vector field over the zero vector field on  $X$ .

We mention the following types of forms on a bundle  $Y \rightarrow X$ :

- *horizontal* forms

$$\phi : Y \rightarrow \bigwedge^r T^*X, \quad \phi = \phi_{\lambda_1 \dots \lambda_r}(y) dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r},$$

- *tangent-valued horizontal* forms

$$\begin{aligned} \phi : Y &\rightarrow \bigwedge_Y^r T^*X \otimes TY, \\ \phi &= dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r} \otimes [\phi_{\lambda_1 \dots \lambda_r}^\mu(y)\partial_\mu + \phi_{\lambda_1 \dots \lambda_r}^i(y)\partial_i], \end{aligned}$$

- *vertical-valued horizontal forms*

$$\phi : Y \rightarrow \overset{r}{\wedge} T^*X \underset{Y}{\otimes} VY, \quad \phi = \phi_{\lambda_1 \dots \lambda_r}^i(y) dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r} \otimes \partial_i,$$

- *pullback-valued forms*

$$Y \rightarrow \overset{r}{\wedge} T^*X \underset{Y}{\otimes} TX, \quad \phi = \phi_{\lambda_1 \dots \lambda_r}^\mu(y) dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r} \otimes \partial_\mu, \quad (2.10)$$

$$Y \rightarrow \overset{r}{\wedge} T^*X \underset{Y}{\otimes} V^*Y, \quad \phi = \phi_{\lambda_1 \dots \lambda_r i}(y) dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r} \otimes \bar{d}y^i. \quad (2.11)$$

Horizontal 1-forms on  $Y$  are called *semi-basic forms*. If such a form is the pullback  $\pi^*\phi$  of a 1-form  $\phi$  on  $X$ , it is said to be a *basic form*. If there is no danger of confusion, we shall denote the pullbacks  $\pi^*\phi$  onto  $Y$  of forms  $\phi$  on  $X$  by the same symbol  $\phi$ .

Vertical-valued horizontal 1-forms

$$\sigma : Y \rightarrow T^*X \underset{Y}{\otimes} VY, \quad \sigma = \sigma_\lambda^i dx^\lambda \otimes \partial_i, \quad (2.12)$$

are termed the *soldering forms*.

Note that the forms (2.10) are not tangent-valued forms, and the forms (2.11) are not exterior forms. The pullbacks

$$\phi = \phi_{\lambda_1 \dots \lambda_r}^\mu(x) dx^{\lambda_1} \wedge \dots \wedge dx^{\lambda_r} \otimes \partial_\mu$$

of tangent-valued forms on  $X$  onto  $Y$  exemplify the pullback-valued forms (2.10).

Horizontal forms of degree  $n = \dim X$  on a bundle  $Y \rightarrow X$  are called the *horizontal densities*. In sequel, we shall exploit the notation

$$\omega = dx^1 \wedge \dots \wedge dx^n, \quad \omega_\lambda = \partial_\lambda \rfloor \omega, \quad \partial_\mu \rfloor \omega_\lambda = \omega_{\mu\lambda}. \quad (2.13)$$

## 2.5 Distributions

Let  $Z$  be an  $m$ -dimensional manifold. A  $k$ -codimensional *smooth distribution*  $\mathcal{T}$  on  $Z$  is defined to be a subbundle of rank  $m - k$  of the tangent bundle  $TZ$ . A smooth distribution  $\mathcal{T}$  is called *involutive* if  $[u, u']$  is a section of  $\mathcal{T}$  whenever  $u$  and  $u'$  are sections of  $\mathcal{T}$ .

Let  $\mathcal{T}$  be a  $k$ -codimensional distribution on  $Z$ . Its annihilator  $\mathcal{T}^*$  is a  $k$ -dimensional subbundle of  $T^*Z$  called the *Pfaffian system*. It means that, on a neighborhood  $U$  of every point  $z \in Z$ , there exist  $k$  linearly independent sections  $\phi_1, \dots, \phi_k$  of  $\mathcal{T}^*$  such that

$$\mathcal{T}_z \rfloor_U = \bigcap_j \text{Ker } \phi_j.$$

Let  $\mathcal{C}(\mathcal{T})$  be the ideal of  $\mathcal{O}(Z)$  generated by sections of  $\mathcal{T}^*$ .

**PROPOSITION 2.3.** A smooth distribution  $\mathcal{T}$  is involutive iff the ideal  $\mathcal{C}(\mathcal{T})$  is *differential*, that is,  $d\mathcal{C}(\mathcal{T}) \subset \mathcal{C}(\mathcal{T})$  [54].  $\square$

**Remark 2.7.** Given an involutive  $k$ -codimensional distribution  $\mathcal{T}$  on  $Z$ , the quotient  $TZ/\mathcal{T}$  is a  $k$ -dimensional vector bundle called the *transversal bundle* of  $\mathcal{T}$ . There is the exact sequence

$$0 \rightarrow \mathcal{T} \hookrightarrow TZ \rightarrow TZ/\mathcal{T} \rightarrow 0. \quad (2.14)$$

Given a bundle  $Y \rightarrow X$ , its vertical tangent bundle  $VY$  exemplifies an involutive distribution on  $Y$ . In this case, the exact sequence (2.14) is just the exact sequence (2.7a).  $\bullet$

A submanifold  $N$  of  $Z$  is called the *integral manifold* of a distribution  $\mathcal{T}$  on  $Z$  if the tangent spaces to  $N$  coincide with the fibres of this distribution at each point of  $N$ .

**THEOREM 2.4.** Let  $\mathcal{T}$  be a smooth involutive distribution on  $Z$ . For any point  $z \in Z$ , there exists a maximal integral manifold of  $\mathcal{T}$  passing through  $z$  [26, 35, 54].  $\square$

In view of this fact, involutive distributions are also called *completely integrable distributions*.

**COROLLARY 2.5.** Every point  $z \in Z$  has an open neighborhood  $U$  which is a domain of a coordinate chart  $(z^1, \dots, z^m)$  such that the restrictions of  $\mathcal{T}$  and  $\mathcal{T}^*$  to  $U$  are generated by the  $m-k$  vector fields  $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{m-k}}$  and the  $k$  Pfaffian forms  $dz^{m-k+1}, \dots, dz^m$  respectively.  $\square$

In particular, it follows that integral manifolds of an involutive distribution constitute a foliation. Recall that a  $k$ -codimensional *foliation* on a  $m$ -dimensional manifold  $Z$  is a partition of  $Z$  into connected leaves  $F_i$  with the following property. Every point of  $Z$  has an open neighborhood  $U$  which is a domain of a coordinate chart  $(z^\alpha)$  such that, for every leaf  $F_i$ , the components  $F_i \cap U$  are described by the equations  $z^{m-k+1} = \text{const.}, \dots, z^m = \text{const.}$  [26, 43]. Note that leaves of a foliation fail to be imbedded submanifolds in general.

**Example 2.8.** Every projection  $\pi : Z \rightarrow X$  defines a foliation whose leaves are the fibres  $\pi^{-1}(x)$ ,  $x \in X$ .  $\bullet$

**Example 2.9.** Every nowhere vanishing vector field  $u$  on a manifold  $Z$  defines a 1-dimensional involutive distribution on  $Z$ . Its integral manifolds are the integral curves of  $u$ . In virtue of the Corollary 2.5, around each point  $z \in Z$ , there exist local coordinates  $(z^1, \dots, z^m)$  of a neighborhood of  $z$  such that  $u$  is given by  $u = \frac{\partial}{\partial z^1}$ .  $\bullet$

## 2.6 First order jet manifolds

Differential operators, differential equations and Lagrangian formalism are conventionally phrased in terms of jet manifolds [8, 30, 41, 47, 45, 50].

Given a bundle  $Y \rightarrow X$ , its first order *jet manifold*  $J^1Y$  comprises the equivalence classes  $j_x^1s$ ,  $x \in X$ , of sections  $s : X \rightarrow Y$  so that sections  $s$  and  $s'$  belong to the same class iff

$$Ts|_{T_xX} = Ts'|_{T_xX}.$$

Roughly speaking, sections  $s, s' \in j_x^1s$  are identified by their values  $s^i(x) = s'^i(x)$  and the values of their partial derivatives  $\partial_\mu s^i(x) = \partial_\mu s'^i(x)$  at the point  $x$  of  $X$ . There are the natural fibrations

$$\pi_1 : J^1Y \ni j_x^1s \mapsto x \in X, \quad \pi_{01} : J^1Y \ni j_x^1s \mapsto s(x) \in Y.$$

Given bundle coordinates  $(x^\lambda, y^i)$  of  $Y$ , the jet manifold  $J^1Y$  is equipped with the adapted coordinates

$$\begin{aligned} (x^\lambda, y^i, y_\lambda^i), \quad (y^i, y_\lambda^i)(j_x^1s) &= (s^i(x), \partial_\lambda s^i(x)), \\ y_\lambda^i &= \frac{\partial x^\mu}{\partial x'^\lambda} (\partial_\mu + y_\mu^j \partial_j) y^i. \end{aligned} \tag{2.15}$$

A glance at (2.15) shows that  $J^1Y \rightarrow Y$  is an affine bundle modelled on the vector bundle

$$T^*X \otimes_Y VY \rightarrow Y. \tag{2.16}$$

**PROPOSITION 2.6.** There exist the canonical bundle monomorphisms

$$\lambda : J^1Y \hookrightarrow T^*X \otimes_Y TY, \quad \lambda = dx^\lambda \otimes d_\lambda = dx^\lambda \otimes (\partial_\lambda + y_\lambda^i \partial_i), \tag{2.17}$$

$$\theta : J^1Y \hookrightarrow T^*Y \otimes_Y VY, \quad \theta = \theta^i \otimes \partial_i = (dy^i - y_\lambda^i dx^\lambda) \otimes \partial_i, \tag{2.18}$$

where  $d_\lambda$  are called the *total derivatives*.  $\square$

**Remark 2.10.** The total derivatives obey the relations

$$d_\lambda \circ d = d \circ d_\lambda, \quad d_\lambda(\phi \wedge \sigma) = d_\lambda \phi \wedge \sigma + \phi \wedge d_\lambda \sigma, \quad \phi, \sigma \in \mathcal{O}(Y).$$

•

The monomorphisms (2.17) and (2.18) enable us to express jets into the tangent-valued forms.

Let  $\Phi : Y \rightarrow Y'$  be a bundle morphism over a diffeomorphism  $f$  of  $X$ . The *jet prolongation* of  $\Phi$  is the morphism

$$J^1\Phi : J^1Y \rightarrow J^1Y', \quad J^1\Phi : j_x^1s \mapsto j_{f(x)}^1(\Phi \circ s \circ f^{-1}).$$

Its coordinate expression is given by (2.15).

Every section  $s$  of a bundle  $Y \rightarrow X$  admits the jet prolongation to the section

$$(J^1s)(x) = j_x^1s, \quad (y^i, y_\lambda^i) \circ J^1s = (s^i(x), \partial_\lambda s^i(x)),$$

of the bundle  $J^1Y \rightarrow X$ . A section of  $J^1Y \rightarrow X$  is called *holonomic* if it is the jet prolongation of a section of  $Y \rightarrow X$ .

Every projectable vector field  $u = u^\lambda \partial_\lambda + u^i \partial_i$  on  $Y \rightarrow X$  has the *jet lift* to the vector field

$$\begin{aligned} \bar{u} &= r \circ J^1u : J^1Y \rightarrow J^1TY \rightarrow TJ^1Y, \\ \boxed{\bar{u} = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda}, \end{aligned} \quad (2.19)$$

on the bundle  $J^1Y \rightarrow X$ . In the definition of  $\bar{u}$ , we use the bundle morphism

$$r : J^1TY \rightarrow TJ^1Y, \quad \dot{y}_\lambda^i \circ r_1 = (\dot{y}^i)_\lambda - y_\mu^i \dot{x}_\lambda^\mu.$$

In particular, there exists the canonical isomorphism

$$VJ^1Y = J^1VY, \quad \dot{y}_\lambda^i = (\dot{y}^i)_\lambda. \quad (2.20)$$

As a consequence, the jet lift (2.19) of a vertical vector field  $u$  on  $Y \rightarrow X$  coincides with its jet prolongation

$$\bar{u} = J^1u = u^i \partial_i + d_\lambda u^i \partial_i^\lambda.$$

If a bundle  $Y \rightarrow X$  is endowed with an algebraic structure, the jet bundle  $J^1Y \rightarrow X$  inherits this algebraic structure due to the jet prolongations of the corresponding morphisms. For instance, if  $Y$  is a vector bundle,  $J^1Y \rightarrow X$  does as well. If  $Y$  is an affine bundle modelled on a vector bundle  $\bar{Y} \rightarrow X$ , then  $J^1Y \rightarrow X$  is an affine bundle modelled on the vector bundle  $J^1\bar{Y} \rightarrow X$ .

The canonical monomorphisms (2.17) and (2.18) determine the *canonical splitting*

$$\boxed{\dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i = \dot{x}^\lambda (\partial_\lambda + y_\lambda^i \partial_i) + (\dot{y}^i - \dot{x}^\lambda y_\lambda^i) \partial_i} \quad (2.21)$$

of the pullback  $J^1Y \times_Y TY$  and the dual splitting

$$\boxed{\dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i = (\dot{x}_\lambda + \dot{y}_i y_\lambda^i) dx^\lambda + \dot{y}_i (dy^i - y_\lambda^i dx^\lambda)} \quad (2.22)$$

of the pullback  $J^1Y \times_Y T^*Y$ . In particular, we get the canonical splitting of a vector field on  $Y$ :

$$u = u^\lambda \partial_\lambda + u^i \partial_i = u_H + u_V = u^\lambda (\partial_\lambda + y_\lambda^i \partial_i) + (u^i - u^\lambda y_\lambda^i) \partial_i. \quad (2.23)$$

## 2.7 Second order jet manifolds

The *repeated jet manifold*  $J^1 J^1 Y$  is defined to be the jet manifold of the bundle  $J^1 Y \rightarrow X$ . It is provided with the adapted coordinates  $(x^\lambda, y^i, y_\lambda^i, y_{(\mu)}^i, y_{\lambda\mu}^i)$ .

There are the projections

$$\pi_{11} : J^1 J^1 Y \rightarrow J^1 Y, \quad y_\lambda^i \circ \pi_{11} = y_\lambda^i, \quad (2.24)$$

$$J^1 \pi_{01} : J^1 J^1 Y \rightarrow J^1 Y, \quad y_\lambda^i \circ J^1 \pi_{01} = y_{(\lambda)}^i. \quad (2.25)$$

They coincide on the *sesquiholonomic subbundle*  $\widehat{J}^2 Y \rightarrow J^1 Y$  of  $J^1 J^1 Y$  which is given by the coordinate conditions  $y_{(\lambda)}^i = y_\lambda^i$ . It is coordinatized by  $(x^\lambda, y^i, y_\lambda^i, y_{\lambda\mu}^i)$ .

The *second order jet manifold*  $J^2 Y$  of a bundle  $Y \rightarrow X$  is the subbundle of  $\widehat{J}^2 Y \rightarrow J^1 Y$  defined by the coordinate conditions  $y_{\lambda\mu}^i = y_{\mu\lambda}^i$ . It is coordinatized by  $(x^\lambda, y^i, y_\lambda^i, y_{\lambda\leq\mu}^i)$  together with the transition functions

$$y_{\lambda\mu}^i = \frac{\partial x^\alpha}{\partial x'^\mu} (\partial_\alpha + y_\alpha^j \partial_j + y_{\nu\alpha}^j \partial_j^\nu) y_{\lambda'}^i.$$

The second order jet manifold  $J^2 Y$  of  $Y$  comprises the equivalence classes  $j_x^2 s$  of sections  $s$  of  $Y \rightarrow X$  such that

$$y_\lambda^i(j_x^2 s) = \partial_\lambda s^i(x), \quad y_{\lambda\mu}^i(j_x^2 s) = \partial_\mu \partial_\lambda s^i(x).$$

In other words, two sections  $s, s' \in j_x^2 s$  are identified by their values and the values of their first and second order derivatives at the point  $x \in X$ .

Let  $\Phi : Y \rightarrow Y'$  be a bundle morphism over a diffeomorphism of  $X$  and  $J^1 \Phi$  its jet prolongation. Let us consider the jet prolongation  $J^1 J^1 \Phi : J^1 J^1 Y \rightarrow J^1 J^1 Y'$  of  $J^1 \Phi$ . Restricted to the second order jet manifold  $J^2 Y$ , the morphism  $J^1 J^1 \Phi$  takes its values in  $J^2 Y'$ . It is called the *second order jet prolongation*  $J^2 \Phi$  of  $\Phi$ .

Similarly, the repeated jet prolongation  $J^1 J^1 s$  of a section  $s$  of  $Y \rightarrow X$  is a section of the bundle  $J^1 J^1 Y \rightarrow X$ . It takes its values into  $J^2 Y$  and defines the following second order jet prolongation of  $s$ :

$$(J^2 s)(x) = (J^1 J^1 s)(x) = j_x^2 s.$$

## 2.8 Ehresmann connections

A *connection*  $\Gamma$  on  $Y$  is usually defined to be a splitting (2.8a) (or (2.8b)) of the exact sequence (2.7a) (or (2.7b)). There are the corresponding decompositions

$$TY = \Gamma(TX) \oplus_Y VY, \quad T^*Y = T^*X \oplus_Y \Gamma(V^*X). \quad (2.26)$$



It is readily observed that the canonical splittings (2.21) – (2.22) of  $TY$  and  $T^*Y$  over the jet bundle  $J^1Y \rightarrow Y$  enable us to recover the splittings (2.26) by means of a section

$$\boxed{\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i(y)\partial_i)}, \quad \Gamma_\lambda^i = \left( \frac{\partial y'^i}{\partial y^j} \Gamma_\mu^j + \frac{\partial y'^i}{\partial x^\mu} \right) \frac{\partial x^\mu}{\partial x'^\lambda}, \quad (2.27)$$

of this jet bundle. Substituting  $\Gamma$  (2.27) into (2.21) – (2.22), we obtain the familiar splittings

$$\begin{aligned} \dot{x}^\lambda \partial_\lambda + \dot{y}^i \partial_i &= \dot{x}^\lambda (\partial_\lambda + \Gamma_\lambda^i \partial_i) + (\dot{y}^i - \dot{x}^\lambda \Gamma_\lambda^i) \partial_i, \\ \dot{x}_\lambda dx^\lambda + \dot{y}_i dy^i &= (\dot{x}_\lambda + \Gamma_\lambda^i \dot{y}_i) dx^\lambda + \dot{y}_i (dy^i - \Gamma_\lambda^i dx^\lambda) \end{aligned} \quad (2.28)$$

corresponding to (2.26).

Hereafter, we follow the notion of a connection on  $Y \rightarrow X$  as a section of the jet bundle  $J^1Y \rightarrow Y$ . It is called the *Ehresmann connection*.

**Example 2.11.** Let  $Y \rightarrow X$  be a vector bundle. A linear connection on  $Y$  reads

$$\Gamma = dx^\lambda \otimes [\partial_\lambda - \Gamma_{j\lambda}^i(x) y^j \partial_i]. \quad (2.29)$$

Let  $Y \rightarrow X$  be an affine bundle modelled on a vector bundle  $\bar{Y} \rightarrow X$ . An affine connection on  $Y$  reads

$$\Gamma = dx^\lambda \otimes [\partial_\lambda + (-\Gamma_{j\lambda}^i(x) y^j + \Gamma_\lambda^i(x)) \partial_i],$$

where  $\bar{\Gamma} = dx^\lambda \otimes [\partial_\lambda - \Gamma_{j\lambda}^i(x) \bar{y}^j \partial_i]$  is a linear connection on  $\bar{Y}$ . ●

Since the affine jet bundle  $J^1Y \rightarrow Y$  is modelled on the vector bundle (2.16), Ehresmann connections on  $Y \rightarrow X$  constitute an affine space modelled on the linear space of soldering forms on  $Y$ . If  $\Gamma$  is a connection and  $\sigma$  is a soldering form (2.12) on  $Y$ , its sum

$$\Gamma + \sigma = dx^\lambda \otimes [\partial_\lambda + (\Gamma_\lambda^i + \sigma_\lambda^i) \partial_i]$$

is a connection on  $Y$ . Conversely, if  $\Gamma$  and  $\Gamma'$  are connections on  $Y$ , then

$$\Gamma - \Gamma' = (\Gamma_\lambda^i - \Gamma'_\lambda^i) dx^\lambda \otimes \partial_i$$

is a soldering form.

We mention the following operations with connections.

(i) Let  $\Gamma$  be a connection on  $Y \rightarrow X$  and  $\Gamma'$  a connection on  $Y' \rightarrow X$ . There exists the *product connection*  $\Gamma \times \Gamma'$  on  $Y \times_X Y'$ .

(ii) Every linear connection  $\Gamma$  on a vector bundle  $Y \rightarrow X$  yields the *dual linear connection*

$$\Gamma_{i\lambda}^* = \Gamma_{i\lambda}^j(x) y_j$$

on the dual vector bundle  $Y^* \rightarrow X$ .

**Example 2.12.** A linear connection  $K$  on the tangent bundle  $TX$  of a manifold  $X$  and the dual connection  $K^*$  to  $K$  on the cotangent bundle  $T^*X$  read

$$\boxed{K_\lambda^\alpha = -K^\alpha_{\nu\lambda}(x)\dot{x}^\nu, \quad K_{\alpha\lambda}^* = K^\nu_{\alpha\lambda}(x)\dot{x}_\nu.} \quad (2.30)$$

•

(iii) Given a connection  $\Gamma$  on  $Y \rightarrow X$ , the vertical tangent morphism  $V\Gamma$  yields the *vertical connection*

$$\boxed{V\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i \frac{\partial}{\partial y^i} + \partial_V \Gamma_\lambda^i \frac{\partial}{\partial y^i}), \quad \partial_V \Gamma_\lambda^i = \dot{y}^j \partial_j \Gamma_\lambda^i,} \quad (2.31)$$

on the bundle  $VY \rightarrow X$  due to the canonical isomorphism (2.20). The dual *covertical connection* on the bundle  $V^*Y \rightarrow X$  reads

$$\boxed{V^*\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i \frac{\partial}{\partial y^i} - \dot{y}_i \partial_j \Gamma_\lambda^i \frac{\partial}{\partial y_j}).} \quad (2.32)$$

(iv) For every connection  $\Gamma$  on  $Y \rightarrow X$ , one can construct its *jet lift*  $J\Gamma$  onto the bundle  $J^1Y \rightarrow X$  as follows. Note that the jet prolongation  $J^1\Gamma$  of the connection  $\Gamma$  on  $Y$  is a section of the repeated jet bundle (2.25), but not of the bundle  $\pi_{11}$  (2.24). Let  $K^*$  be a linear symmetric connection (2.30) on the cotangent bundle  $T^*X$  of  $X$ . There exists the affine bundle morphism

$$\begin{aligned} r_K : J^1J^1Y &\rightarrow J^1J^1Y, & r_K \circ r_K &= \text{Id}_{J^1J^1Y}, \\ (y_\lambda^i, y_{(\mu)}^i, y_{\lambda\mu}^i) \circ r_K &= (y_{(\lambda)}^i, y_\mu^i, y_{\mu\lambda}^i + K^\alpha_{\lambda\mu}(y_\alpha^i - y_{(\alpha)}^i)). \end{aligned}$$

We set

$$\boxed{J\Gamma = r_K \circ J^1\Gamma = dx^\mu \otimes [\partial_\mu + \Gamma_\mu^i \partial_i + (d_\lambda \Gamma_\mu^i - K^\alpha_{\lambda\mu}(y_\alpha^i - \Gamma_\alpha^i)) \partial_i^\lambda].} \quad (2.33)$$

The *curvature* of a connection  $\Gamma$  is given by the horizontal vertical-valued 2-form

$$\begin{aligned} R &= \frac{1}{2} \sum R_{\lambda\mu}^i dx^\lambda \wedge dx^\mu \otimes \partial_i, \\ \boxed{R_{\lambda\mu}^i &= \partial_\lambda \Gamma_\mu^i - \partial_\mu \Gamma_\lambda^i + \Gamma_\lambda^j \partial_j \Gamma_\mu^i - \Gamma_\mu^j \partial_j \Gamma_\lambda^i.} \end{aligned} \quad (2.34)$$

In particular, the curvature of the linear connection (2.29) reads

$$\begin{aligned} R_{\lambda\mu}^i(y) &= -R_{j\lambda\mu}^i(x)y^j, \\ R_{j\lambda\mu}^i &= \partial_\lambda \Gamma_{j\mu}^i - \partial_\mu \Gamma_{j\lambda}^i + \Gamma_{j\mu}^k \Gamma_{k\lambda}^i - \Gamma_{j\lambda}^k \Gamma_{k\mu}^i. \end{aligned}$$

A connection  $\Gamma$  on  $Y \rightarrow X$  yields the first order differential operator

$$D_\Gamma : J^1 Y \rightarrow T^* X \otimes_Y VY, \quad \boxed{D_\Gamma = (y_\lambda^i - \Gamma_\lambda^i) dx^\lambda \otimes \partial_i}, \quad (2.35)$$

called the *covariant differential* relative to  $\Gamma$ . The corresponding *covariant derivative* of a section  $s$  of  $Y$  is

$$\nabla_\Gamma s = D_\Gamma \circ J^1 s = [\partial_\lambda s^i - (\Gamma \circ s)_\lambda^i] dx^\lambda \otimes \partial_i.$$

A local section  $s$  of a  $Y \rightarrow X$  is said to be an *integral section* for a connection  $\Gamma$  on  $Y$  if  $\Gamma \circ s = J^1 s$ , that is,  $\nabla_\Gamma s = 0$ .

**Remark 2.13.** Every connection  $\Gamma$  on the bundle  $Y \rightarrow X$  defines a system of first order differential equations on  $Y$  (in the spirit of [8, 31, 41]) which is an imbedded subbundle  $\Gamma(Y) = \text{Ker } D_\Gamma$  of the jet bundle  $J^1 Y \rightarrow Y$ . It is given by the coordinate relations

$$y_\lambda^i = \Gamma_\lambda^i(y). \quad (2.36)$$

Integral sections for  $\Gamma$  are local solutions of (2.36), and *vice versa*. ●

## 2.9 Curvature-free connections

Every connection  $\Gamma$  on  $Y \rightarrow X$ , by definition, yields the *horizontal distribution*  $\Gamma(TX) \subset TY$  (2.8a) on  $Y$ . It is generated by horizontal lifts

$$\tau_\Gamma = \tau^\lambda (\partial_\lambda + \Gamma_\lambda^i \partial_i)$$

onto  $Y$  of vector fields  $\tau = \tau^\lambda \partial_\lambda$  on  $X$ . The associated Pfaffian system is locally generated by the forms  $(dy^i - \Gamma_\lambda^i dx^\lambda)$ .

**PROPOSITION 2.7.** The horizontal distribution  $\Gamma(TX)$  is involutive iff  $\Gamma$  is a curvature-free connection. □

**Proof.** Straightforward calculations show that

$$[\tau_\Gamma, \tau'_\Gamma] = ([\tau, \tau'])_\Gamma$$

iff the curvature  $R$  (2.34) of  $\Gamma$  vanishes everywhere. ●

**Remark 2.14.** Obviously, not every bundle admits a curvature-free connection. If a principal bundle over a simply connected base (i.e., its first homotopy group is trivial) admits a curvature-free connection, this bundle is trivializable [29]. ●

In virtue of Theorem 2.4, the horizontal distribution defined by a curvature-free connection is completely integrable. The corresponding foliation on  $Y$  is transversal to the foliation defined by the fibration  $\pi : Y \rightarrow X$ . It is called the *horizontal foliation*. Its leaf through a point  $y \in Y$  is defined locally by the integral section  $s_y$  of the connection  $\Gamma$  through  $y$ . Conversely, let  $Y$  admits a horizontal foliation such that, for each point  $y \in Y$ , the leaf of this foliation through  $y$  is locally defined by some section  $s_y$  of  $Y \rightarrow X$  through  $y$ . Then, the map

$$\Gamma : Y \rightarrow J^1Y, \quad \Gamma(y) = j_s^1 s_y, \quad \pi(y) = x.$$

is well defined. This is a curvature-free connection on  $Y$ .

**COROLLARY 2.8.** There is the 1:1 correspondence between the curvature-free connections and the horizontal foliations on a bundle  $Y \rightarrow X$ .  $\square$

Given a horizontal foliation on  $Y \rightarrow X$ , there exists the associated atlas of bundle coordinates  $(x^\lambda, y^i)$  of  $Y$  such that (i) every leaf of this foliation is locally generated by the equations  $y^i = \text{const.}$  and (ii) the transition functions  $y^i \rightarrow y'^i(y^j)$  are independent on the coordinates  $x^\lambda$  of the base  $X$  [26]. It is called the *atlas of constant local trivializations*. Two such atlases are said to be equivalent if their union also is an atlas of constant local trivializations. They are associated with the same horizontal foliation.

**COROLLARY 2.9.** There is the 1:1 correspondence between the curvature-free connections  $\Gamma$  on a bundle  $Y \rightarrow X$  and the equivalence classes of atlases  $\Psi_c$  of constant local trivializations of  $Y$  such that  $\Gamma_\lambda^i = 0$  relative to the coordinates of the corresponding atlas  $\Psi_c$  [9].  $\square$

Connections on a bundle over a 1-dimensional base  $X^1$  are curvature-free connections.

**Example 2.15.** Let  $Y \rightarrow X^1$  be such a bundle ( $X^1 = \mathbf{R}$  or  $X^1 = S^1$ , see Remark 2.1). It is coordinatized by  $(t, y^i)$ , where  $t$  is either the canonical parameter of  $\mathbf{R}$  or the standard local coordinate of  $S^1$  together with the transition functions  $t' = t + \text{const.}$  Relative to this coordinate, the base  $X^1$  is provided with the standard vector field  $\partial_t$  and the standard 1-form  $dt$ . Let  $\Gamma$  be a connection on  $Y \rightarrow X^1$ . In virtue of Proposition 2.7, such a connection defines a horizontal foliation on  $Y \rightarrow X^1$ . Its leaves are the integral curves of the horizontal lift

$$\tau_\Gamma = \partial_t + \Gamma^i \partial_i \tag{2.37}$$

of  $\partial_t$  by  $\Gamma$  (see Example 2.9). The corresponding Pfaffian system is locally generated by the forms  $(dy^i - \Gamma^i dt)$ . There exists an atlas of constant local trivializations  $(t, y^i)$  such that  $\Gamma^i = 0$  and  $\tau_\Gamma = \partial_t$  relative to these coordinates.  $\bullet$

A connection  $\Gamma$  on  $Y \rightarrow X^1$  is called *complete* if the horizontal vector field (2.37) is complete.

**PROPOSITION 2.10.** Every trivialization of  $Y \rightarrow \mathbf{R}$  defines a complete connection. Conversely, every complete connection on  $Y \rightarrow \mathbf{R}$  defines a trivialization  $Y \simeq \mathbf{R} \times M$ . The vector field (2.37) comes to the vector field  $\partial_t$  on  $\mathbf{R} \times M$ .  $\square$

**Proof.** Every trivialization of  $Y \rightarrow \mathbf{R}$  defines a one-parameter group of isomorphisms of  $Y \rightarrow \mathbf{R}$  over  $\text{Id}_{\mathbf{R}}$ , and hence a complete connection. Conversely, let  $\Gamma$  be a complete connection on  $Y \rightarrow \mathbf{R}$ . The vector field  $\tau_\Gamma$  (2.37) is the generator of a 1-parameter group  $G_\Gamma$  which acts freely on  $Y$ . The orbits of this action are of course the integral sections of  $\tau_\Gamma$ . Hence we get a projection  $Y \rightarrow M = Y/G_\Gamma$  which, together with the projection  $Y \rightarrow \mathbf{R}$ , defines a trivialization  $Y \simeq \mathbf{R} \times M$ .  $\bullet$

## 2.10 Composite connections

Let us consider a bundle  $\pi : Y \rightarrow X$  which admits a *composite fibration*

$$Y \xrightarrow{\pi_{Y\Sigma}} \Sigma \xrightarrow{\pi_{\Sigma X}} X, \quad (2.38)$$

where  $Y \rightarrow \Sigma$  and  $\Sigma \rightarrow X$  are bundles. It is equipped with the bundle coordinates  $(x^\lambda, \sigma^m, y^i)$  together with the transition functions

$$x^\lambda \rightarrow x'^\lambda(x^\mu), \quad \sigma^m \rightarrow \sigma'^m(x^\mu, \sigma^n), \quad y^i \rightarrow y'^i(x^\mu, \sigma^n, y^j),$$

where  $(x^\mu, \sigma^m)$  are bundle coordinates of  $\Sigma \rightarrow X$ .

**Example 2.16.** We have the composite bundles

$$TY \rightarrow Y \rightarrow X, \quad VY \rightarrow Y \rightarrow X, \quad J^1Y \rightarrow Y \rightarrow X.$$

$\bullet$

Let

$$A = dx^\lambda \otimes (\partial_\lambda + A_\lambda^i \partial_i) + d\sigma^m \otimes (\partial_m + A_m^i \partial_i) \quad (2.39)$$

be a connection on the bundle  $Y \rightarrow \Sigma$  and

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^m \partial_m)$$

a connection on the bundle  $\Sigma \rightarrow X$ . Given a vector field  $\tau$  on  $X$ , let us consider its horizontal lift  $\tau_\Gamma$  onto  $\Sigma$  by  $\Gamma$  and then the horizontal lift  $(\tau_\Gamma)_A$  of  $\tau_\Gamma$  onto  $Y$  by the connection (2.39).

PROPOSITION 2.11. There exists the connection

$$\boxed{\gamma = dx^\lambda \otimes [\partial_\lambda + \Gamma_\lambda^m \partial_m + (A_m^i \Gamma_\lambda^m + A_\lambda^i) \partial_i].} \quad (2.40)$$

on  $Y \rightarrow X$  such that the horizontal lift  $\tau_\gamma$  onto  $Y$  of any vector field  $\tau$  on  $X$  consists with the above mentioned lift  $(\tau_\Gamma)_A$  [45, 48]. It is called the *composite connection*.  $\square$

Given a composite bundle  $Y$  (2.38), the exact sequence

$$0 \rightarrow VY_\Sigma \hookrightarrow VY \rightarrow Y \times_\Sigma V\Sigma \rightarrow 0$$

over  $Y$  take place, where  $VY_\Sigma$  is the vertical tangent bundle of  $Y \rightarrow \Sigma$ . Every connection (2.39) on the bundle  $Y \rightarrow \Sigma$  yields the splitting

$$\begin{aligned} VY &= VY_\Sigma \oplus_Y (Y \times_\Sigma V\Sigma), \\ y^i \partial_i + \dot{\sigma}^m \partial_m &= (\dot{y}^i - A_m^i \dot{\sigma}^m) \partial_i + \dot{\sigma}^m (\partial_m + A_m^i \partial_i). \end{aligned}$$

Due to this splitting, one can construct the first order differential operator

$$\begin{aligned} \widetilde{D} &= \text{pr}_1 \circ D_\gamma : J^1 Y \rightarrow T^* X \otimes_Y VY \rightarrow T^* X \otimes_Y VY_\Sigma, \\ \boxed{\widetilde{D} = dx^\lambda \otimes (y_\lambda^i - A_\lambda^i - A_m^i \sigma_\lambda^m) \partial_i,} \end{aligned} \quad (2.41)$$

on the composite manifold  $Y$ , where  $D_\gamma$  is the covariant differential (2.35) relative to the composite connection (2.40). We call  $\widetilde{D}$  the *vertical covariant differential*.

### 3 Symplectic geometry

This Section aims to recall the basic notions of symplectic geometry which we shall need in sequel [1, 2, 35, 52].

#### 3.1 Jacobi structure

Let  $Z$  be a manifold. The *Jacobi bracket* (or the *Jacobi structure*) on  $Z$  is defined to be a bilinear map

$$S(Z) \times S(Z) \ni (f, g) \rightarrow \{f, g\} \in S(Z),$$

where  $S(Z)$  is the linear space of smooth functions on  $Z$ , which satisfies the following conditions:

- (A1)  $\{g, f\} = -\{f, g\}$  (skew-symmetry),
- (A2)  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$  (Jacobi identity),
- (A3) the support of  $\{f, g\}$  is contained in the intersection of the supports of  $f$  and  $g$ .

PROPOSITION 3.1. Every Jacobi bracket on a manifold  $Z$  is uniquely defined in accordance with the relation

$$\boxed{\{f, g\} = w(df, dg) + u](fdg - gdf)]} \quad (3.1)$$

by a vector field  $u$  and a bivector field  $w$  on  $Z$  such that

$$\mathbf{L}_u w = 0, \quad [w, w] = 2u \wedge w \quad (3.2)$$

[28, 37, 38].  $\square$

Taking  $w = 0$ , every vector field  $u$  on a manifold  $Z$  defines the Jacobi bracket (3.1). The relations (3.2) are obviously satisfied.

The Jacobi bracket (3.1) with  $u = 0$  is said to be a *Poisson bracket* (see Section 3.3). Contact forms on an odd-dimensional manifold generate Jacobi brackets which are not the Poisson ones (see next Section).

#### 3.2 Contact forms

DEFINITION 3.2. Given a  $(2m + 1)$ -dimensional manifold  $Z$ , a *contact form* on  $Z$  is defined to be a Pfaffian form  $\theta$  such that the form  $\theta \wedge (d\theta)^m \neq 0$  everywhere on  $Z$ . The pair  $(Z, \theta)$  is called the *contact manifold*.  $\square$

Note that a manifold  $Z$  equipped with a contact form  $\theta$  is orientable, and  $\theta \wedge (d\theta)^m$  is a volume element.

The assertion below is a variant of the well-known Darboux's theorem [35].

**THEOREM 3.3.** Let  $(Z, \theta)$  be a  $(2m + 1)$ -dimensional contact manifold. Every point  $z$  of  $Z$  has an open neighborhood  $U$  which is the domain of a coordinate chart  $(z^0, \dots, z^{2m})$  such that the contact form  $\theta$  has the local expression

$$\theta = dz^0 - \sum_{i=1}^m z^{m+i} dz^i$$

on  $U$ . These coordinates are called *Darboux's coordinates*.  $\square$

If  $\theta$  is a contact form, its differential  $d\theta$  is a presymplectic form of rank  $2m$  (see Definition 3.10).

**PROPOSITION 3.4.** Let  $\theta$  be a contact form on  $Z$ . There exists the unique nowhere vanishing vector field  $E$  on  $Z$  such that

$$\boxed{E \lrcorner \theta = 1, \quad E \lrcorner d\theta = 0.}$$

It is called the *Reeb vector field* of  $\theta$  [35].  $\square$

Relative to Darboux's coordinates, the Reeb vector field reads  $E = \partial_0$ .

**PROPOSITION 3.5.** Every contact form  $\theta$  on an odd-dimensional manifold  $Z$  yields the associated Jacobi structure on  $Z$ . It is defined by the Reeb vector field  $E$  of  $\theta$  and by the bivector field  $w$  such that

$$w^\sharp \phi \lrcorner \theta = 0, \quad w^\sharp \phi \lrcorner d\theta = -(\phi - (E \lrcorner \phi)\theta) \quad (3.3)$$

for every  $\phi \in \mathcal{O}^1(Z)$  [38].  $\square$

Relative to Darboux's coordinates, the Jacobi structure (3.3) reads

$$\{f, g\} = \sum_{i=1}^m (\partial_{m+i} g \partial_i f - \partial_{m+i} f \partial_i g) + (\tilde{g} \partial_0 f - \tilde{f} \partial_0 g),$$

where

$$\tilde{f} = \sum_{i=1}^m z^{m+i} \partial_{m+i} f + f, \quad \tilde{g} = \sum_{i=1}^m z^{m+i} \partial_{m+i} g + g.$$



### 3.3 Poisson structure

According to (3.2), a bivector field  $w$  on a manifold  $Z$  provides a *Poisson structure* if it obeys the condition

$$[w, w] = 0, \quad w^{\mu\lambda_1} \partial_\mu w^{\lambda_2\lambda_3} + w^{\mu\lambda_2} \partial_\mu w^{\lambda_3\lambda_1} + w^{\mu\lambda_3} \partial_\mu w^{\lambda_1\lambda_2} = 0.$$

It is called the *Poisson bivector* (see Example 2.4). A manifold  $Z$  equipped with a Poisson bivector  $w$  is called the *Poisson manifold*  $(Z, w)$ .

Besides the conditions (A1 – A3), the *Poisson bracket*

$$\boxed{\{f, g\} = w(df, dg)}$$

satisfies also the Leibniz rule

$$\{h, fg\} = \{h, f\}g + f\{h, g\}. \quad (3.4)$$

A Poisson structure defined by a Poisson bivector  $w$  is said to be *regular* if the associated morphism  $w^\sharp : T^*Z \rightarrow TZ$  (2.4) has a constant rank. Hereafter, by a Poisson structure is meant a regular Poisson structure.

A Poisson structure is called *nondegenerate* if  $w^\sharp$  is an isomorphism. If the Poisson structure  $w$  is nondegenerate, it induces the symplectic form  $\Omega$  on  $Z$  defined by the coordinate relation  $\Omega_{\alpha\beta} = (w^{-1})_{\beta\alpha}$ , and *vice versa* (see Proposition 3.11). A nondegenerate Poisson structure can exist only on an even-dimensional manifold (see next Section).

Note that there are no pullback or push-forward operations of Poisson structures by manifold morphisms in general. The following assertion deals with Poisson projections, whereas Theorem 3.8 is concerned with Poisson injections.

**PROPOSITION 3.6.** Let  $(Z, w)$  be a Poisson manifold and  $\pi : Z \rightarrow Y$  a projection. The following properties are equivalent:

(i) for every pair  $(f, g)$  of functions on  $Y$  and for each point  $y \in Y$ , the restriction of the function  $\{f \circ \pi, g \circ \pi\}$  to the fibre  $\pi^{-1}(y)$  is constant;

(ii) there exists a Poisson structure on  $Y$  for which  $\pi$  is a Poisson morphism.

If such a Poisson structure exists, it is unique [35].  $\square$

**DEFINITION 3.7.** Given a function  $f$  on a Poisson manifold  $(Z, w)$ , the image

$$\boxed{\vartheta_f = w^\sharp df, \quad \vartheta_f = w^{\mu\nu} \partial_\mu f \partial_\nu,}$$

of its differential  $df$  by the morphism  $w^\sharp$  is called the *Hamiltonian vector field* of  $f$ .  $\square$

The Hamiltonian vector field  $\vartheta_f$ , by definition, obeys the relation

$$\boxed{\vartheta_f]dg = \{f, g\}} \quad (3.5)$$

for any function  $g$  on  $Z$ . It is easy to see that

$$[\vartheta_f, \vartheta_g] = \vartheta_{\{f, g\}}. \quad (3.6)$$

This relation provides the set of Hamiltonian vector fields with a Lie algebra structure. Using (3.4) and (3.6), one can show that

$$(\mathbf{L}_{\vartheta_h} w)(df, dg) = \vartheta_h]d\{f, g\} - \{\vartheta_h]df, g\} - \{f, \vartheta_h]dg\} = 0.$$

It follows that a Hamiltonian vector field generates a (local) 1-parameter group of isomorphisms of the Poisson manifold  $(Z, w)$ .

The values of all Hamiltonian vector fields at all points of  $Z$  constitute the *characteristic distribution of the Poisson manifold*  $(Z, w)$ . In virtue of the relation (3.6), this distribution is involutive. We have the following theorem.

**THEOREM 3.8.** The characteristic distribution of a Poisson manifold  $(Z, w)$  is completely integrable. The Poisson structure induces the symplectic structures on leaves of the corresponding foliation of  $Z$  [52]. It is called the *symplectic foliation*.  $\square$

Of course, the symplectic foliation has the adapted coordinates described in Corollary 2.5. Moreover, one can choose these coordinates in such a way to bring the Poisson bracket in the following canonical form [52, 55].

**PROPOSITION 3.9.** For any point  $z$  of a Poisson manifold, there are local coordinates  $(z^1, \dots, z^r, y^1, \dots, y^k, p_1, \dots, p_k)$  near  $z$  such that

$$\{y^i, y^j\} = \{p_i, p_j\} = \{y^i, z^\alpha\} = \{p_i, z^\alpha\} = \{z^\beta, z^\alpha\} = 0, \quad \{p_i, y^j\} = \delta_i^j. \quad (3.7)$$

$\square$

### 3.4 Symplectic structure

**DEFINITION 3.10.** A 2-form  $\Omega$  on a manifold  $Z$  is called *presymplectic* if it is closed. A presymplectic form  $\Omega$  is said to be *symplectic* if it is nondegenerate (see Example 2.3).  $\square$

A manifold  $Z$  equipped with a symplectic [presymplectic] form is said to be a *symplectic* [presymplectic] manifold.

PROPOSITION 3.11. On every even-dimensional manifold  $Z$ , there is the 1:1 correspondence between the symplectic forms  $\Omega$  and the Poisson bivectors  $w$  in accordance with the equalities

$$\boxed{w(\phi, \sigma) = \Omega(w^\sharp \phi, w^\sharp \sigma), \quad \Omega(\vartheta, \nu) = w(\Omega^\flat \vartheta, \Omega^\flat \nu),} \quad \phi, \sigma \in \mathcal{O}^1(Z), \quad \vartheta, \nu \in \mathcal{V}^1(Z),$$

(see relations (2.2) and (2.4)) [36].  $\square$

In particular, the notion of a Hamiltonian vector field may also be introduced on symplectic manifolds.

DEFINITION 3.12. A vector field  $\vartheta$  on a symplectic manifold  $(Z, \Omega)$  is said to be *locally Hamiltonian* [*Hamiltonian*] if the form  $\vartheta \lrcorner \Omega$  is closed [exact].  $\square$

As an immediate consequence of this definition, we observe that:

(i) a vector field  $\vartheta$  is locally Hamiltonian iff it is an infinitesimal symplectomorphism, that is,

$$\boxed{\mathbf{L}_\vartheta \Omega = d(\vartheta \lrcorner \Omega) = 0;}$$

(ii) a vector field  $\vartheta$  is Hamiltonian iff it is a Hamiltonian vector field in accordance with Definition 3.7, i.e.  $\vartheta = \vartheta_f$ , where

$$\boxed{df = -\vartheta_f \lrcorner \Omega, \quad \vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.}$$

Note that Definition 3.12 of locally Hamiltonian and Hamiltonian vector fields applies also to presymplectic manifolds.

**Example 3.1.** Let  $M$  be a manifold coordinatized by  $(y^i)$  and let  $T^*M$  be its cotangent bundle provided with the holonomic coordinates  $(y^i, p_i = \dot{y}_i)$ . The cotangent bundle  $T^*M$  is a well-known symplectic manifold equipped with the *canonical symplectic form*

$$\boxed{\Omega = dp_i \wedge dy^i} \tag{3.8}$$

which is the differential of the canonical Liouville form

$$\boxed{\theta = p_i dy^i} \tag{3.9}$$

on  $T^*M$ . These forms are canonical in the sense that the expressions (3.8) and (3.9) are maintained under arbitrary transformations of the coordinates  $y^i$  and the corresponding holonomic transformations of the coordinates  $p_i$ . Furthermore, for every closed 2-form  $\phi$  on  $M$ , the form  $\Omega + \phi$  is also a symplectic form on  $T^*M$ .  $\bullet$

The canonical symplectic form (3.8) plays a fundamental role in view of Darboux's theorem [35].

**THEOREM 3.13.** Let  $(Z, \Omega)$  be a symplectic manifold. Every point  $x$  of  $Z$  has an open neighborhood  $U$  which is the domain of a coordinate chart  $(y^1, \dots, y^n, p_1, \dots, p_n)$  such that the symplectic form  $\Omega$  has the local expression (3.8) on  $U$ . Such coordinates are called *canonical*.  $\square$

**Proof.** It is an immediate consequence of Proposition 3.9 and Proposition 3.11.  $\bullet$

**Remark 3.2.** Canonical coordinates on the manifold  $T^*M$  are not adapted to the fibration  $T^*M \rightarrow M$  in general. For instance, the local coordinates  $(y'^i = -p_i, p'_i = y^i)$  on  $T^*M$  also are canonical.  $\bullet$

## 4 Polysymplectic geometry

We consider first order Lagrangian and Hamiltonian formalisms on a bundle  $Y \rightarrow X$  over an  $n$ -dimensional base manifold  $X$  [11, 18, 25, 27, 46, 48].

### 4.1 Lagrangian formalism

Let  $Y \rightarrow X$  be a bundle coordinatized by  $(x^\lambda, y^i)$ . In jet terms, a *first order Lagrangian* is defined to be a horizontal density  $L = \mathcal{L}\omega$  on the jet manifold  $J^1Y$  (see the notation (2.13)). The jet manifold  $J^1Y$  plays the role of the finite-dimensional configuration space of sections of  $Y \rightarrow X$ . We shall use the notation

$$\pi_i^\lambda = \partial_i^\lambda \mathcal{L}, \quad \tilde{\pi} = \mathcal{L} - \pi_i^\lambda y_\lambda^i.$$

We base our consideration on the *first variational formula* which provides the canonical decomposition of the Lie derivatives of Lagrangians along projectable vector fields in accordance with the variational task [3, 16, 19, 32, 49].

Let  $u = u^\lambda \partial_\lambda + u^i \partial_i$  be a projectable vector field on  $Y \rightarrow X$  and  $\bar{u}$  its jet lift (2.19). Given a Lagrangian density  $L$ , we have the following canonical decomposition of the Lie derivative of  $L$  along  $u$ :

$$\boxed{\mathbf{L}_u L \equiv u_V \rfloor \mathcal{E}_L + h_0 d(u \rfloor \Xi_L),} \quad (4.1)$$

where  $u_V$  is the vertical part (2.23) of  $u$ ,

$$h_0 : dy^i \mapsto y_\lambda^i dx^\lambda, \quad dy_\mu^i \mapsto y_{\mu\lambda}^i dx^\lambda,$$

is the operator of horizontalization,

$$\boxed{\mathcal{E}_L = (\partial_i - d_\lambda \partial_i^\lambda) \mathcal{L} dy^i \wedge \omega} \quad (4.2)$$

is the *Euler–Lagrange operator*, and  $\Xi_L$  is some *Lepagean equivalent* of  $L$  on  $J^1Y$ .

We restrict our consideration to the *Poincaré–Cartan form*

$$\boxed{\Xi_L = \pi_i^\lambda dy^i \wedge \omega_\lambda + \tilde{\pi} \omega.} \quad (4.3)$$

This is the only Lepagean equivalent which has the partner in the framework of the Hamiltonian formalism (see Section 4.7). Moreover, if  $n = 1$ , this is the unique Lepagean equivalent of a Lagrangian.

The kernel  $\text{Ker } \mathcal{E}_L \subset J^2Y$  of the Euler–Lagrange operator (4.2) defines the system of second order *Euler–Lagrange equations*

$$\boxed{(\partial_i - d_\lambda \partial_i^\lambda) \mathcal{L} = 0} \quad (4.4)$$

on the bundle  $Y \rightarrow X$ . On sections  $s$  of  $Y \rightarrow X$ , these equations read

$$\partial_i \mathcal{L} - (\partial_\lambda + \partial_\lambda s^j \partial_j + \partial_\lambda \partial_\mu s^j \partial_j^\mu) \partial_i^\lambda \mathcal{L} = 0. \quad (4.5)$$

**Remark 4.1.** Note that different Lagrangians  $L$  and  $L'$  lead to the same Euler–Lagrange operator iff

$$L' = L + h_0(\epsilon), \quad (4.6)$$

where  $\epsilon$  is a closed  $n$ -form on  $Y$  [32]. Any closed form  $\epsilon$  on  $Y$  is a Lepagean form. Let  $L$  be a Lagrangian and  $\rho_L$  its Lepagean equivalent. Then, the Lepagean form  $\rho_L + \epsilon$  is the Lepagean equivalent of the Lagrangian (4.6). •

## 4.2 Legendre morphisms

Every first order Lagrangian  $L$  yields the *Legendre morphism*  $\hat{L}$  of the jet manifolds  $J^1 Y$  to the *Legendre manifold*

$$\Pi = V^* Y \underset{Y}{\wedge}^{n-1} (T^* X) = V^* Y \underset{Y}{\wedge}^n (T^* X) \underset{Y}{\otimes} TX \quad (4.7)$$

which plays the role of the finite-dimensional phase space of sections of  $Y \rightarrow X$ . Given the bundle coordinates  $(x^\lambda, y^i)$  of  $Y \rightarrow X$ , the Legendre bundle (4.7) is coordinatized by  $(x^\lambda, y^i, p_i^\lambda)$ , where  $p_i^\lambda$  are the holonomic coordinates with the transition functions

$$p_i'^\lambda = \det\left(\frac{\partial x^\epsilon}{\partial x'^\nu}\right) \frac{\partial y^j}{\partial y'^i} \frac{\partial x'^\lambda}{\partial x^\mu} p_j^\mu. \quad (4.8)$$

Relative to these coordinates, the Legendre morphism  $\hat{L}$  reads

$$p_i^\mu \circ \hat{L} = \pi_i^\mu. \quad (4.9)$$

The Poincaré–Cartan form  $\Xi_L$  (4.3) defines a morphism  $\hat{\Xi}_L$  of the jet manifold  $J^1 Y$  to the *homogeneous Legendre manifold*

$$Z = T^* Y \wedge (\wedge^{n-1} T^* X) \quad (4.10)$$

provided with the holonomic coordinates  $(x^\lambda, y^i, p_i^\lambda, p)$  with the transition functions (4.8) and

$$p' = \det\left(\frac{\partial x^\epsilon}{\partial x'^\nu}\right) \left(p - \frac{\partial y^j}{\partial y'^i} \frac{\partial y'^i}{\partial x^\mu} p_j^\mu\right). \quad (4.11)$$

Relative to these coordinates, the morphism  $\widehat{\Xi}_L$  reads

$$(p_i^\mu, p) \circ \widehat{\Xi}_L = (\pi_i^\mu, \widetilde{\pi}). \quad (4.12)$$

A glance at the expression (4.11) shows that  $Z \rightarrow \Pi$  is a 1-dimensional affine bundle. We have the exact sequence

$$0 \rightarrow \overset{n}{\wedge} T^*X \hookrightarrow Z \rightarrow \Pi \rightarrow 0.$$

The homogeneous Legendre manifold (4.10) is equipped with the canonical  $n$ -form

$$\Xi = p\omega + p_i^\lambda dy^i \wedge \omega_\lambda. \quad (4.13)$$

Its coordinate expression (4.13) is maintained under holonomic coordinate transformations (4.8) and (4.11). The Poincaré–Cartan form  $\Xi_L$  (4.3) is the pullback of  $\Xi$  by the morphism  $\widehat{\Xi}_L$  (4.12).

### 4.3 Polysymplectic structure

The Legendre manifold (4.7) possesses the *canonical polysymplectic form*

$$\mathbf{\Lambda} = dp_i^\lambda \wedge dy^i \wedge \omega \otimes \partial_\lambda \quad (4.14)$$

whose coordinate expression (4.14) is maintained under holonomic coordinate transformations (4.8). It is a pullback-valued form of the type (2.10).

**Remark 4.2.** The polysymplectic form (4.14) can be introduced in different ways. The Legendre manifold  $\Pi$  is equipped also with the *generalized Liouville form*

$$\Theta = -p_i^\lambda dy^i \wedge \omega \otimes \partial_\lambda. \quad (4.15)$$

Since (4.15) is a pullback-valued form, one can not act on  $\Theta$  by the exterior differential in order to recover the polysymplectic form  $\mathbf{\Lambda}$  (4.14). At the same time,  $\mathbf{\Lambda}$  is the unique form which obeys the relation

$$\mathbf{\Lambda} \rfloor \phi = -d(\Theta \rfloor \phi)$$

for any Pfaffian form  $\phi$  on  $X$ . •

Given the atlas of holonomic coordinates  $(x^\lambda, y^i, p_i^\lambda)$ , let us examine the coordinate transformations between these coordinates and any coordinate atlas adapted to the bundle  $\Pi \rightarrow X$  which keep invariant the coordinate form (4.14) of  $\mathbf{\Lambda}$ . They will be called the *polysymplectic canonical coordinate transformations*.

We find that, since  $y^i$  and  $p_i^\lambda$  parameterize spaces of different dimensions if  $n > 1$ , polysymplectic canonical coordinate transformations have a simpler structure than that of the symplectic ( $n = 1$ ) ones (see Remark 3.2). Precisely they are compatible with the fibration  $\Pi \rightarrow Y$  and are exhausted by the holonomic coordinate transformations (4.8) and the translations

$$p_i'^\lambda = p_i^\lambda + r_i^\lambda(y), \quad \partial_j r_i^\lambda(y) = \partial_i r_j^\lambda(y). \quad (4.16)$$

Hereafter, we consider only holonomic coordinates  $(x^\lambda, y^i, p_i^\lambda)$  of  $\Pi$ .

#### 4.4 Hamiltonian connections

Let  $J^1\Pi$  be the jet manifold of the bundle  $\Pi \rightarrow X$ . It is provided with the adapted coordinates  $(x^\lambda, y^i, p_i^\lambda, y_\mu^i, p_{i\mu}^\lambda)$ .

**DEFINITION 4.1.** By analogy with the notion of a Hamiltonian vector field (see Definition 3.12), a connection

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma_\lambda^i \partial_i + \gamma_{i\lambda}^\mu \partial_\mu^i)$$

on the bundle  $\Pi \rightarrow X$  is said to be *locally Hamiltonian* [*Hamiltonian*] if the exterior form  $\gamma]\Lambda$  is closed [exact].  $\square$

It is readily observed that a connection  $\gamma$  is locally Hamiltonian iff it obeys the conditions

$$\partial_\lambda^i \gamma_\mu^j - \partial_\mu^j \gamma_\lambda^i = 0, \quad \partial_i \gamma_{j\mu}^\mu - \partial_j \gamma_{i\mu}^\mu = 0, \quad \partial_j \gamma_\lambda^i + \partial_\lambda^i \gamma_{j\mu}^\mu = 0. \quad (4.17)$$

**Example 4.3.** Given a linear symmetric connection  $K$  (2.30) on  $T^*X$ , every connection  $\Gamma$  on the bundle  $Y \rightarrow X$  gives rise to the connection

$$\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma_\lambda^i \partial_i + (-\partial_j \Gamma_\lambda^i p_i^\mu - K^\mu_{\nu\lambda} p_j^\nu + K^\alpha_{\alpha\lambda} p_j^\mu) \partial_\mu^j] \quad (4.18)$$

on  $\Pi \rightarrow X$ . It is easy to see that  $\tilde{\Gamma}]\Lambda = 0$  and, consequently, the connection (4.18) is a locally Hamiltonian connection. Actually  $\tilde{\Gamma}$  appears to be a Hamiltonian connection (see Example 4.4).  $\bullet$



## 4.5 Hamiltonian forms

DEFINITION 4.2. A  $n$ -form  $H$  on the Legendre bundle  $\Pi$  is called a *general Hamiltonian form* if there exists a Hamiltonian connection such that

$$\boxed{\gamma \rfloor \Lambda = dH.}$$

□

Unless otherwise stated, general Hamiltonian forms will be considered modulo closed forms.

PROPOSITION 4.3. Let  $H$  be a general Hamiltonian form. For any horizontal density  $\widetilde{H} = \widetilde{\mathcal{H}}\omega$  on the bundle  $\Pi \rightarrow X$ , the form  $H - \widetilde{H}$  is a Hamiltonian form. □

The following example shows that general Hamiltonian forms on  $\Pi$  always exist.

**Example 4.4.** Let  $\Gamma$  and  $K$  be as in Example 4.3. Then  $\widetilde{\Gamma}$  (4.18) is a Hamiltonian connection for the general Hamiltonian form

$$\boxed{H_\Gamma = \Gamma \rfloor \Theta = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma_\lambda^i(y) \omega,}$$

where  $\Theta$  is the generalized Liouville form (4.15). ●

DEFINITION 4.4. A general Hamiltonian form  $H$  on  $\Pi$  is said to be *Hamiltonian* if it has the splitting

$$\boxed{H = H_\Gamma - \widetilde{H}_\Gamma = p_i^\lambda dy^i \wedge \omega_\lambda - (p_i^\lambda \Gamma_\lambda^i + \widetilde{\mathcal{H}}_\Gamma) \omega = p_i^\lambda dy^i \wedge \omega_\lambda - \mathcal{H} \omega} \quad (4.19)$$

modulo closed forms, where  $\Gamma$  is a connection on  $Y$  and  $\widetilde{H}_\Gamma$  is a horizontal density. □

This splitting is preserved under the holonomic coordinate transformations (4.8), but not under translations (4.16).

PROPOSITION 4.5. There is the 1:1 correspondence between the Hamiltonian forms  $H$  and the sections  $h$  of the bundle  $Z \rightarrow \Pi$ . We have

$$\boxed{H = h^* \Xi,}$$

where  $\Xi$  is the canonical form (4.13) on  $Z$  □

**Proof.** It is an immediate consequence of the expression (4.19) for Hamiltonian forms. ●

By a *momentum morphism* we shall mean any bundle morphism

$$\Phi : \Pi \xrightarrow{Y} J^1 Y, \quad \Phi = dx^\lambda \otimes (\partial_\lambda + \Phi_\lambda^i \partial_i). \quad (4.20)$$

For instance, let  $\Gamma$  be a connection on the bundle  $Y \rightarrow X$ . Then, the composition  $\Gamma \circ \pi_{\Pi Y}$  is a momentum morphism. Conversely, every momentum morphism  $\Phi$  defines the associated connection  $\Gamma_\Phi = \Phi \circ \widehat{0}$  on  $Y \rightarrow X$ , where  $\widehat{0}$  is the global zero section of  $\Pi \rightarrow Y$ .

PROPOSITION 4.6. Every Hamiltonian form  $H$  (4.19) on the Legendre manifold  $\Pi$  yields the associated momentum morphism

$$\widehat{H} : \Pi \rightarrow J^1Y, \quad (x^\lambda, y^i, y_\lambda^i) \circ \widehat{H} = (x^\lambda, y^i, \partial_\lambda^i \mathcal{H}), \quad (4.21)$$

and the associated connection  $\Gamma_H = \widehat{H} \circ \widehat{0}$  on  $Y \rightarrow X$ . Conversely, every momentum morphism (4.20) defines the Hamiltonian form

$$H_\Phi = \Phi] \Theta = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Phi_\lambda^i \omega.$$

□

Given a Hamiltonian form  $H$  (4.19), there are the algebraic conditions

$$\gamma_\lambda^i = \partial_\lambda^i \mathcal{H}, \quad \gamma_{i\lambda}^\lambda = -\partial_i \mathcal{H}$$

for a Hamiltonian connection  $\gamma$  to be associated with a given Hamiltonian form  $H$ . It should be emphasized that, if  $n > 1$ , there exist different Hamiltonian connections for the same Hamiltonian form in general.

Let a Hamiltonian connection  $\gamma$  associated with a Hamiltonian form  $H$  have an integral section  $r$  of  $\Pi \rightarrow X$ , that is,  $\gamma \circ r = J^1r$ . Then  $r$  satisfies the system of first order differential equations

$$y_\lambda^i = \partial_\lambda^i \mathcal{H}, \quad (4.22a)$$

$$p_{i\lambda}^\lambda = -\partial_i \mathcal{H} \quad (4.22b)$$

on  $\Pi$ . They are called the *Hamilton equations*. It is readily observed that, if  $r$  is a solution of the Hamilton equations (4.22a) – (4.22b), it obeys the relations

$$J^1(\pi_{\Pi Y} \circ r) = \widehat{H} \circ r.$$

## 4.6 Hamiltonian and Lagrangian formalisms

We now turn to the relations between the Lagrangian formalism and the polysymplectic Hamiltonian formalism. Let  $L$  be a Lagrangian and  $Q = \widehat{L}(J^1Y)$ . We shall call  $Q$  the Lagrangian constraint space.

A Hamiltonian form  $H$  is said to be *associated* with  $L$  if it obeys the conditions

$$\widehat{L} \circ \widehat{H}|_Q = \text{Id}_Q, \quad , \boxed{p_i^\mu = \partial_i^\mu \mathcal{L}(x^\lambda, y^j, \partial_\lambda^j \mathcal{H}(p)), \quad p \in Q,} \quad (4.23a)$$

$$H_{\widehat{H}} - H = L \circ \widehat{H}, \quad \boxed{p_i^\mu \partial_\mu^i \mathcal{H} - \mathcal{H} \equiv \mathcal{L}(x^\lambda, y^j, \partial_\lambda^j \mathcal{H}).} \quad (4.23b)$$

It should be emphasized that there may be different Hamiltonian forms associated with  $L$  in general. We restrict our consideration to Lagrangians which are *semiregular*, that is, the preimage  $\widehat{L}^{-1}(q)$  of each point  $q \in Q$  is a connected submanifold of  $J^1Y$ .

All Hamiltonian forms associated with a semiregular Lagrangian  $L$  coincide with each other on the Lagrangian constraint space  $Q$ , and the Poincaré–Cartan form  $\Xi_L$  (4.3) is the pullback

$$\Xi_L = \widehat{L}^* H, \quad \boxed{\pi_i^\lambda y_\lambda^i - \mathcal{L} \equiv \mathcal{H}(x^\mu, y^i, \pi_i^\lambda),} \quad (4.24)$$

of any such a Hamiltonian form  $H$  by the Legendre morphism  $\widehat{L}$ . In this case, the following relation between solutions of the Euler–Lagrange equations and solutions of the Hamilton equations takes place [44, 56].

**PROPOSITION 4.7.** Let a section  $r$  of  $\Pi \rightarrow X$  be a solution of the Hamilton equations (4.22a) – (4.22b) for a Hamiltonian form  $H$  associated with a semiregular Lagrangian  $L$ . If  $r$  lives on the Lagrangian constraint space  $Q$ , the section  $s = \pi_{\Pi Y} \circ r$  of  $Y \rightarrow X$  satisfies the Euler–Lagrange equations (4.5) for  $L$ . Conversely, given a semiregular Lagrangian  $L$ , let  $s$  be a solution of the corresponding Euler–Lagrange equations. Let  $H$  be a Hamiltonian form associated with  $L$  such that

$$\widehat{H} \circ \widehat{L} \circ J^1 s = J^1 s. \quad (4.25)$$

Then, the section  $r = \widehat{L} \circ J^1 s$  of  $\Pi \rightarrow X$  is a solution of the Hamilton equations (4.22a) – (4.22b) for  $H$ .  $\square$

We say that a family of Hamiltonian forms  $H$  associated with a semiregular Lagrangian  $L$  is *complete* if, for each solution  $s$  of the Euler–Lagrange equations, there exists a solution  $r$  of the Hamilton equations for some Hamiltonian form  $H$  of this family so that

$$r = \widehat{L} \circ J^1 s, \quad s = \pi_{\Pi Y} \circ r. \quad (4.26)$$

**Example 4.5.** In case of a *hyperregular* Lagrangian  $L$  (i.e., the Legendre morphism  $\widehat{L}$  is a diffeomorphism), the Lagrangian formalism and the polysymplectic Hamiltonian formalism are equivalent. There exists the unique Hamiltonian form

$$H = H_{\widehat{L}^{-1}} + L \circ \widehat{L}^{-1}$$

associated with  $L$ . The corresponding momentum morphism (4.21) is the diffeomorphism  $\widehat{H} = \widehat{L}^{-1}$ . As a consequence, there is the 1:1 correspondence between the solutions of the Euler–Lagrange equations for  $L$  and the Hamilton equations for  $H$ . In case of a *regular* Lagrangian  $L$  (i.e.,  $\widehat{L}$  is a local diffeomorphism), the Lagrangian constraint space  $Q$  is an open submanifold of the Legendre manifold  $\Pi$ . If a regular Lagrangian density is also semiregular, the associated Legendre morphism is a diffeomorphism of  $J^1Y$  onto  $Q$  and, on  $Q$ , we can recover all results true for hyperregular Lagrangians. ●

**Remark 4.6.** Given a Hamiltonian form  $H$  (4.19) on the Legendre manifold  $\Pi$  (4.7), let us consider the Lagrangian

$$\boxed{L_H = (p_i^\lambda y_\lambda^i - \mathcal{H})\omega} \quad (4.27)$$

on the configuration space  $J^1\Pi$  coordinatized by  $(x^\lambda, y^i, p_i^\mu, y_\lambda^i, p_{i\lambda}^\mu)$ . This Lagrangian does not depend on the coordinates  $p_{i\lambda}^\mu$ . It is readily observed that the Poincaré–Cartan form  $\Xi_{L_H}$  (4.3) of the Lagrangian (4.27) consists with the Hamiltonian form  $H$  and the Euler–Lagrange equations (4.4) for  $L_H$  recover the Hamilton equations (4.22a) – (4.22b) for  $H$ . ●

## 4.7 Vertical extension of the polysymplectic formalism

In time-dependent mechanics, the machinery that we present below provides the the way to maintain the form (4.19) of Hamiltonian forms under canonical transformations. By analogy with the BRS generalization of mechanics [22, 23], it represents also a first step toward the BRS quantization of the polysymplectic Hamiltonian formalism.

Given a bundle  $Y \rightarrow X$ , let us consider its vertical tangent bundle  $VY$  coordinatized by  $(x^\lambda, y^i, \dot{y}^i)$ . We show that the Hamiltonian formalism for sections of  $Y \rightarrow X$  is naturally extended to the Hamiltonian formalism for sections of  $VY \rightarrow X$ .

The Legendre manifolds (4.7) corresponding to  $VY \rightarrow X$  is

$$\Pi_{VY} = V^*VY \underset{VY}{\wedge} (\overset{n-1}{\wedge} T^*X).$$

It is coordinatized by  $(x^\lambda, y^i, \dot{y}^i, q_i^\lambda, v_i^\lambda)$ .

**PROPOSITION 4.8.** In virtue of the bundle isomorphism (2.9), there exists the bundle isomorphism

$$\boxed{\Pi_{VY} \underset{VY}{=} V\Pi}, \quad q_i^\lambda \longleftrightarrow \dot{p}_i^\lambda, \quad v_i^\lambda \longleftrightarrow p_i^\lambda, \quad (4.28)$$

where  $(x^\lambda, y^i, p_i^\lambda, \dot{y}^i, \dot{p}_i^\lambda)$  are the coordinates of  $V\Pi$ . □

We shall utilize the compact notation

$$\dot{\partial}_i = \frac{\partial}{\partial \dot{y}^i}, \quad \dot{\partial}_\lambda = \frac{\partial}{\partial \dot{p}_i^\lambda}. \quad (4.29)$$

Recall also the notation  $\partial_V$  (2.6).

One can develop the Hamiltonian formalism on  $\Pi_{VY}$  by analogy with that on  $\Pi$ . The manifold  $V\Pi$  is equipped with the canonical polysymplectic form

$$\boxed{\Lambda_V = [d\dot{p}_i^\lambda \wedge dy^i + dp_i^\lambda \wedge d\dot{y}^i] \wedge \omega \otimes \partial_\lambda.} \quad (4.30)$$

Its coordinate expression is maintained under holonomic transformations of the composite bundle  $V\Pi \rightarrow \Pi \rightarrow Y$ .

**PROPOSITION 4.9.** Let  $\gamma$  be a Hamiltonian connection on  $\Pi$  associated with a Hamiltonian form  $H$  (4.19). Then, the vertical connection  $V\gamma$  (2.31) is a Hamiltonian connection associated with the Hamiltonian form

$$\boxed{H_V = (\dot{p}_i^\lambda dy^i - \dot{y}^i dp_i^\lambda) \wedge \omega_\lambda - \mathcal{H}_V, \quad \mathcal{H}_V = \partial_V \mathcal{H} = (\dot{y}^i \partial_i + \dot{p}_i^\lambda \partial_\lambda) \mathcal{H}.} \quad (4.31)$$

□

**Proof.** It is easily justified that, given a Hamiltonian connection

$$\gamma = dx^\mu \otimes (\partial_\mu + \gamma_\mu^i \partial_i + \gamma_{i\mu}^\lambda \partial_\lambda), \quad \gamma_\mu^i = \partial_\mu^i \mathcal{H}, \quad \gamma_{i\lambda}^\lambda = -\partial_i \mathcal{H},$$

the vertical connection

$$V\gamma = dx^\mu \otimes [\partial_\mu + \gamma_\mu^i \partial_i + \gamma_{i\mu}^\lambda \partial_\lambda + \partial_V \gamma_\mu^i \dot{\partial}_i + \partial_V \gamma_{i\mu}^\lambda \dot{\partial}_\lambda]$$

obeys the Hamilton equations for the Hamiltonian form (4.31):

$$\begin{aligned} \gamma_\mu^i &= \dot{\partial}_\mu^i \mathcal{H}_V = \partial_\mu^i \mathcal{H}, \\ \gamma_{i\lambda}^\lambda &= -\dot{\partial}_i \mathcal{H}_V = -\partial_i \mathcal{H}, \\ \dot{\gamma}_\mu^i &= \partial_\mu^i \mathcal{H}_V = \partial_V \partial_\mu^i \mathcal{H}, \\ \dot{\gamma}_{i\lambda}^\lambda &= -\partial_i \mathcal{H}_V = -\partial_V \partial_i \mathcal{H}. \end{aligned}$$

●

In particular, if there is the splitting  $\mathcal{H} = p_i^\lambda \Gamma_\lambda^i + \widetilde{\mathcal{H}}$  relative to some connection  $\Gamma$  on  $Y \rightarrow X$ , then we have the splitting

$$\mathcal{H}_V = \dot{p}_i^\lambda \Gamma_\lambda^i - \dot{y}^j (-p_i^\lambda \partial_j \Gamma_\lambda^i) + \partial_V \widetilde{\mathcal{H}}$$

with respect to the lift  $\tilde{\Gamma}$  (4.18) of  $\Gamma$  onto  $\Pi \rightarrow X$ .

Note that the Hamiltonian form  $H_V$  (4.31) can be obtained also in the following way. Given the homogeneous Legendre manifold  $Z$  (4.10), let us consider the vertical tangent bundle  $VZ$  of  $Z \rightarrow X$  coordinatized by  $(x^\lambda, y^i, p_i^\lambda, p, \dot{y}^i, \dot{p}_i^\lambda, \dot{p})$ . It is provided with the canonical form

$$\Xi_V = \dot{p}\omega + \dot{p}_i^\lambda dy^i \wedge \omega_\lambda - \dot{y}^i dp_i^\lambda \wedge \omega_\lambda$$

whose expression is maintained under holonomic coordinate transformations. Note that one can utilize also the form  $\Xi_V + d(\dot{y}^i p_i^\lambda) \wedge \omega_\lambda$  since the form  $d(\dot{y}^i p_i^\lambda) \wedge \omega_\lambda$  is well-behaved.

Put  $H = h^*\Xi$ , where  $h$  is a section of the bundle  $Z \rightarrow \Pi$ . Then, we have

$$H_V = (Vh)^*\Xi_V,$$

where  $Vh : V\Pi \rightarrow VZ$  is the vertical tangent morphism to  $h$ .

We now turn to the vertical extension of the Lagrangian formalism on  $J^1Y$  onto the configuration space  $VJ^1Y = J^1VY$  coordinatized by  $(x^\lambda, y^i, y_\lambda^i, \dot{y}^i, \dot{y}_\lambda^i)$ . Given a Lagrangian  $L$  on  $J^1Y$ , let us consider the Lagrangian

$$L_V = \text{pr}_2 \circ VL : VJ^1Y \rightarrow \overset{n}{\wedge} T^*X, \quad \mathcal{L}_V = \partial_V \mathcal{L} = (\dot{y}^i \partial_i + \dot{y}_\lambda^i \partial_i^\lambda) \mathcal{L}, \quad (4.32)$$

on  $VJ^1Y$ . It is readily observed that the variational derivative  $\delta_i \mathcal{L}_V = \delta_i \mathcal{L}$  recovers the Euler–Lagrange equations (4.4).

The Lagrangian (4.32) yields the Legendre morphism

$$\begin{aligned} \widehat{L}_V &= V\widehat{L} : VJ^1Y \xrightarrow{VY} V\Pi, \\ p_i^\lambda &= \partial_i^\lambda \mathcal{L}_V = \pi_i^\lambda, \quad \dot{p}_i^\lambda = \partial_V \pi_i^\lambda. \end{aligned} \quad (4.33)$$

Conversely, given a Hamiltonian form  $H_V$  (4.31) on  $V\Pi$ , there is the momentum morphism

$$\begin{aligned} \widehat{H}_V &= V\widehat{H} : V\Pi \xrightarrow{VY} VJ^1Y, \\ y_\lambda^i &= \partial_\lambda^i \mathcal{H}_V = \partial_\lambda^i \mathcal{H}, \quad \dot{y}_\lambda^i = \partial_V \partial_\lambda^i \mathcal{H}. \end{aligned}$$

Let us consider the relation between the Hamiltonian form  $H_V$  and the Lagrangian  $L_V$  if the Hamiltonian form  $H$  is associated with the Lagrangian  $L$ .

**PROPOSITION 4.10.** The Legendre morphism (4.33) is a surjection of  $VJ^1Y$  onto  $VQ$ .  $\square$

**Proof.** One can show that the equations

$$\dot{p}_i^\lambda = (\dot{y}^i \partial_i + \dot{y}_\lambda^i \partial_i^\lambda) \pi_i^\lambda$$

are equivalent to the equations

$$(\dot{p}_i^\lambda \partial_\lambda^i + \dot{y}^i \partial_i) \rfloor d[p_i^\mu - \partial_i^\mu \mathcal{L}(x^\lambda, y^j, \partial_\lambda^j \mathcal{H}(p))] = 0$$

characterizing tangent vectors to the fibres of the Lagrangian constraint bundle  $Q$ . ●

Moreover,  $VQ$  appears to be the image of  $\hat{L}_V$  restricted to  $\hat{H}(Q)$ . It follows that a relation similar to (4.23a) takes place. At the same time, a relation similar to (4.23b) holds only on the constraint space  $Q$ . Let a Hamiltonian form  $H$  be associated with a semiregular Lagrangian  $L$ . Then, the Hamiltonian form  $H_V$  and the Lagrangian  $L_V$  (which fails to be semiregular in general) satisfy the relation similar to (4.24) on  $\hat{H}(Q)$ .

## 5 Time-dependent Hamiltonian mechanics

To describe time-dependent mechanical systems, let us consider a bundle  $Y \rightarrow X$  with a  $m$ -dimensional standard fibre  $M$  over a 1-dimensional base  $X$ . It is provided with bundle coordinates  $(t, y^i)$ . Observe that:

- (i) the jet manifold  $J^1Y$  is modelled on the vertical tangent bundle  $VY$  of  $Y$ ;
- (ii) the Legendre bundle  $\Pi$  (4.7) over  $Y$  is the vertical cotangent bundle  $V^*Y$  of  $Y$  coordinatized by  $(t, y^i, p_i)$ ;
- (iii) the homogeneous Legendre bundle  $Z$  (4.10) over  $Y$  is the cotangent bundle  $T^*Y$  of  $Y$  coordinatized by  $(t, y^i, p, p_i)$ .

**Remark 5.1.** If the base manifold is contractible, i.e.  $X = \mathbf{R}$ , the bundle  $Y \rightarrow X$  is trivializable. Given a trivialization

$$Y \simeq \mathbf{R} \times M, \quad (5.1)$$

we have the corresponding splittings

$$\begin{aligned} J^1Y &\simeq \mathbf{R} \times TM \\ \Pi &\simeq \mathbf{R} \times T^*M. \end{aligned} \quad (5.2)$$

•

### 5.1 $n = 1$ Reduction of the polysymplectic formalism

The phase space  $\Pi = V^*Y$ . It is provided with the holonomic coordinates  $(t, y^i, p_i)$  possessing the transition functions

$$p'_i = \frac{\partial y^j}{\partial y'^i} p_j. \quad (5.3)$$

The Legendre manifold  $\Pi$  admits the canonical form  $\Lambda$  (4.14) which reads

$$\boxed{\Lambda = dp_i \wedge dy^i \wedge dt \otimes \partial_t.} \quad (5.4)$$

As a particular case of the polysymplectic machinery of the previous Section, we say that a connection

$$\gamma = dt \otimes (\partial_t + \gamma^i \partial_i + \gamma_i \partial^i)$$



on the bundle  $\Pi \rightarrow X$  is locally Hamiltonian [Hamiltonian] if the exterior form  $\gamma \rfloor \mathbf{\Lambda}$  is closed [exact]. A connection  $\gamma$  is locally Hamiltonian iff it obeys the conditions (4.17) which now take the form

$$\partial^i \gamma^j - \partial^j \gamma^i = 0, \quad \partial_i \gamma_j - \partial_j \gamma_i = 0, \quad \partial_j \gamma^i + \partial^i \gamma_j = 0.$$

As in Example 4.3, we observe that every connection  $\Gamma = dt \otimes (\partial_t + \Gamma^i \partial_i)$  on the bundle  $Y \rightarrow X$  gives rise to the Hamiltonian connection

$$\boxed{\tilde{\Gamma} = dt \otimes (\partial_t + \Gamma^i \partial_i - \partial_j \Gamma^i p_i \partial^j)} \quad (5.5)$$

on  $\Pi \rightarrow X$  which consists with the covertical connection  $V^* \Gamma$  (2.32). The corresponding Hamiltonian form is

$$H_\Gamma = p_i dy^i - p_i \Gamma^i dt. \quad (5.6)$$

Let  $H$  be a Hamiltonian form (4.19) on  $\Pi = V^* Y$ . It reads

$$\boxed{H = p_i dy^i - \mathcal{H} dt = p_i dy^i - p_i \Gamma^i dt - \tilde{\mathcal{H}}_\Gamma dt.} \quad (5.7)$$

We call  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  in the decomposition (5.7) the *Hamiltonian* and the *Hamilton function* respectively. Let  $\gamma$  be a Hamiltonian connection on  $\Pi \rightarrow X$  associated with the Hamiltonian form (5.7). It satisfies the relations

$$\begin{aligned} \gamma \rfloor \mathbf{\Lambda} &= dp_i \wedge dy^i + \gamma_i dy^i \wedge dt - \gamma^i dp_i \wedge dt = dH, \\ \gamma^i &= \partial^i \mathcal{H}, \quad \gamma_i = -\partial_i \mathcal{H}. \end{aligned} \quad (5.8)$$

A glance at the equations (5.8) shows that, in the case of mechanics, one and only one Hamiltonian connection is associated with a given Hamiltonian form.

In accordance with Remark 2.13, every connection  $\gamma$  on  $\Pi \rightarrow X$  yields the system of first order differential equations (2.36) which reads

$$y_t^i = \gamma^i, \quad p_{it} = \gamma_i. \quad (5.9)$$

They are called the *evolution equations*. If  $\gamma$  is a Hamiltonian connection associated with the Hamiltonian form  $H$  (5.7), the evolution equations (5.9) come to the Hamilton equations

$$y_t^i = \partial^i \mathcal{H}, \quad (5.10a)$$

$$p_{it} = -\partial_i \mathcal{H}. \quad (5.10b)$$

Note that, once a trivialization (5.2) is chosen, the Hamiltonian form (5.7) yields the well-known Poincaré–Cartan integral invariant [35]. At the same time, the splitting (5.7) is

not maintained under canonical transformations (see Section 5.4). This fact calls into play the general Hamiltonian forms (see Proposition 5.10).

Another well-known ingredient in time-dependent mechanics is the horizontal lift

$$\boxed{\tau_H = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i} \quad (5.11)$$

onto  $\Pi$  of the standard vector field  $\partial_t$  on  $X$  by means of a Hamiltonian connection  $\gamma$  associated with a Hamiltonian form  $H$  (5.7). It is a nowhere vanishing vector field on  $\Pi$  which obeys the relations

$$\tau_H \rfloor H = p_i \partial^i \mathcal{H} - \mathcal{H}, \quad \tau_H \rfloor dH = 0. \quad (5.12)$$

We call  $\tau_H$  (5.11) the *horizontal Hamiltonian vector field* of the Hamiltonian form  $H$ .

**Remark 5.2.** Every connection  $\gamma$  on a bundle  $\Pi \rightarrow X$  is a curvature-free connection (see Example 2.15). In virtue of Proposition 2.7, such a connection defines a horizontal foliation on  $\Pi \rightarrow X$ . Its leaves are the integral curves of the horizontal lift

$$\tau_\gamma = \partial_t + \gamma^i \partial_i + \gamma_i \partial^i \quad (5.13)$$

of  $\partial_t$  by  $\gamma$ . The corresponding Pfaffian system is locally generated by the forms  $(dy^i - \gamma^i dt)$  and  $(dp_i - \gamma_i dt)$ . •

It follows that every Hamiltonian connection and, accordingly, every Hamiltonian form defines the corresponding Hamiltonian foliation on  $\Pi$ . Its leaves are integral curves of the horizontal Hamiltonian vector field (5.11). One can think of these integral curves as being the generalized solutions of the Hamilton equations (5.10a) and (5.10b) (in accordance with the definition of generalized solutions given in [31]). They locally coincide with the integral sections of the Hamiltonian connection  $\gamma$ .

Given a function  $f$  on  $\Pi$ , we have the *Hamiltonian evolution equation*

$$\boxed{\tau_H \rfloor df = d_{Ht} f = (\partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i) f} \quad (5.14)$$

relative to the Hamiltonian  $\mathcal{H}$ . On solutions  $r$  of the Hamilton equations, (5.14) is equal to the total time derivative of the function  $f$ :

$$r^* d_{Ht} f = \frac{d}{dt} (f \circ r).$$

The goal is to write the Hamiltonian evolution equation (5.14) in the terms of a Poisson bracket.

## 5.2 Canonical Poisson structure

Let us consider the homogeneous phase space  $Z = T^*Y$ . The canonical form  $\Xi$  (4.13) on it comes to the canonical Liouville form

$$\boxed{\Xi = p dt + p_i dy^i.} \quad (5.15)$$

Its exterior differential is the canonical symplectic form

$$\boxed{d\Xi = dp \wedge dt + dp_i \wedge dy^i.} \quad (5.16)$$

The corresponding Poisson bracket on the space  $S(Z)$  of functions on  $Z$  reads

$$\boxed{\{f, g\} = \partial^p f \partial_t g - \partial^p g \partial_t f + \partial^i f \partial_i g - \partial^i g \partial_i f.} \quad (5.17)$$

Let us consider the subspace of  $S(Z)$  which consists of the pullbacks of functions on  $\Pi$  by the projection  $Z \rightarrow \Pi$ . It is easily observed that this subspace is closed under the Poisson bracket (5.17). Then, according to Proposition 3.6, one can show that the canonical Poisson structure (5.17) on  $Z$  induces the canonical Poisson structure

$$\boxed{\{f, g\}_V = \partial^i f \partial_i g - \partial^i g \partial_i f} \quad (5.18)$$

on  $\Pi$  by the projection  $Z \rightarrow \Pi$ . The corresponding bivector on  $\Pi$  is vertical with respect to the projection  $\Pi \rightarrow X$ . It reads

$$w^{ij} = 0, \quad w_{ij} = 0, \quad w^i_j = 1.$$

Since the rank of  $w$  is constant, the Poisson structure (5.18) is regular.

The Poisson structure (5.18) is obviously degenerate. It defines the symplectic foliation on  $\Pi$  which coincides with the fibration  $\Pi \rightarrow X$ . The Hamiltonian vector fields associated with the Poisson bracket (5.18) are the vertical vector fields on  $\Pi \rightarrow X$ . The Hamiltonian vector field  $\vartheta_f$  of a function  $f$  is defined by the relation (3.5):

$$\{f, g\}_V = \vartheta_f \lrcorner dg, \quad g \in S(\Pi).$$

It reads

$$\boxed{\vartheta_f = \partial^i f \partial_i - \partial_i f \partial^i.} \quad (5.19)$$

Note that the bundle coordinates  $(t, y^i, p_i)$  of  $\Pi$  are exactly the canonical coordinates (3.7) for the Poisson structure (5.18). In particular, the symplectic forms on the fibres of  $\Pi \rightarrow X$  are the pullbacks

$$\Omega_x = dp_i \wedge dy^i$$

of the canonical symplectic form on the standard fibre  $T^*M$  of the bundle  $\Pi \rightarrow X$  with respect to morphisms of trivialization.

The Poisson structure (5.18) on  $\Pi$  can be introduced in a different way [10]. There exists the canonical closed 3-form

$$\boxed{\Omega = dp^i \wedge dy_i \wedge dt,} \quad (5.20)$$

on the Legendre manifold  $\Pi$ . With this form, every function  $f$  on  $\Pi$  defines a vertical vector field  $\vartheta_f$  on the bundle  $\Pi \rightarrow X$  by the relation

$$\boxed{\vartheta_f \lrcorner \Omega = df \wedge dt.}$$

Then, the Poisson bracket (5.18) is given by condition

$$\vartheta_g \lrcorner \vartheta_f \lrcorner \Omega = \{f, g\}_V dt. \quad (5.21)$$

The canonical forms  $\mathbf{A}$  (5.4) and  $\Omega$  (5.20) on  $\Pi$  can be considered on the same footing as follows.

**PROPOSITION 5.1.** Let  $u$  be a vector field on  $\Pi$  projected onto the standard vector field  $\partial_t$  on  $X$ . This vector field obeys the relation

$$\mathbf{L}_u \Omega = d(u \lrcorner \Omega) = 0 \quad (5.22)$$

iff it is the horizontal lift  $\tau_\gamma$  (5.13) of  $\partial_t$  onto  $\Pi$  by means of a locally Hamiltonian connection  $\gamma$  on  $\Pi \rightarrow X$ . In particular,  $\tau_\gamma \lrcorner \Omega = dH$  if  $\gamma$  is a Hamiltonian connection and  $H$  is the corresponding general Hamiltonian form.  $\square$

**PROPOSITION 5.2.** If  $\gamma$  is a Hamiltonian connection associated with a Hamiltonian form  $H$  and  $\vartheta_f$  is the Hamiltonian vector field (5.19), then  $\gamma + \vartheta_f dt$  is a Hamiltonian connection associated with the Hamiltonian form  $H + f dt$ .  $\square$

Given a Hamiltonian connection  $\gamma$  for a Hamiltonian form  $H$ , let us consider its splitting

$$H = H_0 + \widetilde{\mathcal{H}} dt, \quad \gamma = \gamma_0 + \vartheta dt, \quad (5.23)$$

where  $H_0$  is some Hamiltonian form,  $\gamma_0$  is the Hamiltonian connection for  $H_0$ , and  $\vartheta$  is the Hamiltonian vector field of the function  $\widetilde{\mathcal{H}}$ . One can bring the Hamiltonian evolution equation (5.14) relative to  $H$  into the form compatible with the splitting (5.23). It reads

$$d_{Ht} f = d_{H_0 t} f + \{\widetilde{\mathcal{H}}, f\}_V = \partial_t f + (\gamma_0^i \partial_i + \gamma_{0i} \partial^i) f + \{\widetilde{\mathcal{H}}, f\}_V. \quad (5.24)$$

A glance at this expression shows that Hamiltonian evolution equations in time-dependent mechanics do not reduce to the Poisson brackets.

⌋ This fact becomes relevant to the quantization problem. The second term on the right-hand side of the equation (5.24) remains classical. ⌋

In this context, the main problem is to express the Hamiltonian evolution equation of a classical system in terms of the Poisson bracket. Then, one can bring this Hamiltonian evolution equation into the operator evolution equation under quantization. A glance at the expression (5.24) shows that this is possible only with respect to the splitting (5.23), where the connection  $\gamma_0$  is brought into zero by a canonical coordinate transformation (see Section 5.4).

From physical viewpoint, the splitting (5.23) has a meaning if the connection  $\gamma_0$  characterizes a reference frame (see Section 5.6).

### 5.3 Presymplectic and contact structures

Besides the canonical Poisson structure, there is no other canonical structure on the phase space  $\Pi = V^*Y$  of time-dependent mechanics in general. At the same time, there are structures on  $\Pi$  specified by the choice of a Hamiltonian form  $H$ .

In virtue of Proposition 4.5, every Hamiltonian form  $H$  on the phase space  $\Pi$  is the pullback  $H = h^*\Xi$  of the Liouville form (5.15) by a section  $h$  of the bundle  $Z \rightarrow \Pi$ . Accordingly, its differential

$$dH = (dp_i + \partial_i \mathcal{H} dt) \wedge (dy^i - \partial^i \mathcal{H} dt)$$

is the pullback  $h^*d\Xi$  of the symplectic form (5.16). It is a presymplectic form of the constant rank  $2m$  since the form

$$(dH)^m = (dp_i \wedge dy^i)^m - m(dp_i \wedge dy^i)^{m-1} \wedge d\mathcal{H} \wedge dt \quad (5.25)$$

is obviously nowhere vanishing. However, this presymplectic structure does not introduce any essentially new object because the corresponding Hamiltonian vector fields are proportional to the horizontal Hamiltonian vector field  $\tau_H$  (5.11). At the same time, a Hamiltonian form (5.7) satisfying certain conditions is a contact form which defines a nondegenerate Jacobi structure on  $\Pi$  as follows.

**PROPOSITION 5.3.** A Hamiltonian form (5.7) is a contact form if the density

$$[\mathcal{H}] = p_i \partial^i \mathcal{H} - \mathcal{H}$$

nowhere vanishes [35].  $\square$

**Proof.** Since the horizontal Hamiltonian vector field  $\tau_H$  (5.12) is nowhere vanishing, the condition  $H \wedge (dH)^m \neq 0$  is equivalent to the condition

$$\tau_H \rfloor (H \wedge (dH)^m) = (\tau_H \rfloor H)(dH)^m = [\mathcal{H}]dH^m \neq 0$$

and, since the form  $(dH)^m$  (5.25) is nowhere vanishing, the result follows. ●

**Remark 5.3.** In order to make  $[\mathcal{H}]$  nowhere vanishing, one may add some exact form (e.g.,  $cdt$ ,  $c = \text{const.}$ ) to  $H$ . For instance, the Hamiltonian form  $H_\Gamma$  (5.6) is not a contact form because  $[\mathcal{H}] = 0$ , but the equivalent form  $H_\Gamma - dt$ , where  $[\mathcal{H}] = 1$ , is it. ●

Given a Hamiltonian form  $H$ , let  $[\mathcal{H}]$  be nowhere vanishing so that the form  $H$  is a contact form. The corresponding Reeb vector field reads

$$E_H = [\mathcal{H}]^{-1} \tau_H. \quad (5.26)$$

In virtue of Proposition 3.5, this form has the associated Jacobi bracket defined by the Reeb vector field (5.26) and by the bivector field  $w_H$  derived from the relations (3.3). We find

$$\begin{aligned} w_H(\phi, \sigma) &= w_H^\sharp \phi \rfloor \sigma = \phi^i \sigma_i + p_i \sigma^i E_H \rfloor \phi - [\phi \longleftrightarrow \sigma], \\ w_H^\sharp \phi &= -p_i \phi^i [\mathcal{H}]^{-1} \partial_t + (\phi^i - p_j \phi^j [\mathcal{H}]^{-1} \partial^i \mathcal{H}) \partial_i + \\ &\quad (-\phi_i + [\mathcal{H}]^{-1} (p_j \phi^j \partial_i \mathcal{H} + p_i \tau_H \rfloor \phi)) \partial^i, \end{aligned}$$

where  $\phi$  and  $\sigma$  are Pfaffian forms on  $\Pi$ . The corresponding Jacobi bracket (3.1) reads

$$\{f, g\}_H = \{f, g\}_V + [\mathcal{H}]^{-1} ([g] d_{Ht} f - [f] d_{Ht} g), \quad (5.27)$$

where  $\{f, g\}_V$  is the canonical Poisson bracket (5.18) and

$$[f] = p_i \partial^i f - f, \quad [g] = p_i \partial^i g - g.$$

In particular, let  $H$  have the splitting (5.23). We find

$$\{\widetilde{\mathcal{H}}, g\}_H = [\mathcal{H}]^{-1} ([g] d_{H_0t} \widetilde{\mathcal{H}} - [\widetilde{\mathcal{H}}] d_{H_0t} g). \quad (5.28)$$

Given a contact Hamiltonian form  $H$ , one can consider also the Jacobi bracket defined by the Reeb vector field  $E_H$  (5.26) alone. It reads

$$\{f, g\}_E = [\mathcal{H}]^{-1} (f d_{Ht} g - g d_{Ht} f). \quad (5.29)$$

Given the splitting (5.23), we find

$$\{\widetilde{\mathcal{H}}, g\}_E = [\mathcal{H}]^{-1} (\widetilde{\mathcal{H}} d_{Ht} g - g d_{H_0t} \widetilde{\mathcal{H}}). \quad (5.30)$$

A glance at the expressions (5.28) and (5.30) shows that the Jacobi brackets (5.27) and (5.29) have no advantage over the Poisson bracket (5.18) in order to write the Hamiltonian evolution equation (5.14).

## 5.4 Canonical transformations

Up to now, we have followed the polysymplectic Hamiltonian formalism and have considered transformations which keep the fibration  $\Pi \rightarrow Y$ . Let us now examine canonical transformations of time-dependent mechanics which are not compatible with this fibration. Remind that the base  $X$  is not transformed.

**DEFINITION 5.4.** Given an atlas  $\Psi = \{\psi_\xi\}$  of the bundle  $\Pi \rightarrow X$ , the bundle coordinates  $(t, y^i, p_i)$ , where

$$y^i(p) = (y^i \circ \text{pr}_2 \circ \psi_\xi)(p), \quad p_i(p) = (p_i \circ \text{pr}_2 \circ \psi_\xi)(p), \quad p \in \Pi,$$

are said to be the *canonical coordinates* if, in these coordinates, the form  $\mathbf{\Lambda}$  and equivalently the form  $\mathbf{\Omega}$  are given by the canonical expressions (5.4) and (5.20).  $\square$

The *canonical coordinate transformations* satisfy the relations

$$\begin{aligned} \frac{\partial p'_i}{\partial p_j} \frac{\partial y'^i}{\partial p_k} - \frac{\partial p'_i}{\partial p_k} \frac{\partial y'^i}{\partial p_j} &= 0, \\ \frac{\partial p'_i}{\partial y^j} \frac{\partial y'^i}{\partial y^k} - \frac{\partial p'_i}{\partial y^k} \frac{\partial y'^i}{\partial y^j} &= 0, \\ \frac{\partial p'_i}{\partial p_j} \frac{\partial y'^i}{\partial y^k} - \frac{\partial p'_i}{\partial y^j} \frac{\partial y'^i}{\partial p_k} &= \delta_j^k. \end{aligned} \tag{5.31}$$

By definition, the holonomic coordinates of  $\Pi = V^*Y$  are obviously canonical coordinates.

**DEFINITION 5.5.** By a *canonical transformation (morphism)* is meant an isomorphism  $\rho$  of the bundle  $\Pi \rightarrow X$  over  $X$  such that any atlas  $\Psi$  of holonomic coordinates  $(t, y^i, p_i)$  of  $\Pi$  and the atlas  $\Psi \circ \rho^{-1}$  (2.1) of the coordinates

$$(t, y'^i = y^i \circ \rho^{-1}, p'_i = p_i \circ \rho^{-1})$$

are related by the canonical coordinate transformations (5.31).  $\square$

The equivalent coordinate-free definition of a canonical morphism is the following.

**DEFINITION 5.6.** A canonical morphism is an isomorphism  $\rho$  of the bundle  $\Pi \rightarrow X$  over  $X$  which preserve the canonical form  $\mathbf{\Omega}$  (5.20), that is,  $\rho^*\mathbf{\Omega} = \mathbf{\Omega}$ .  $\square$

It is easily observed that canonical morphisms preserve the canonical Poisson structure (5.18) on  $\Pi$ , that is,

$$\{f \circ \rho, g \circ \rho\}_V = (\{f, g\}_V) \circ \rho.$$

PROPOSITION 5.7. Canonical morphisms send Hamiltonian connections to Hamiltonian connections.  $\square$

**Proof.** The proof is based on the relation  $T\rho(\tau_\gamma) = \tau_{\rho(\gamma)}$ , where  $\gamma$  is a connection on  $\Pi \rightarrow X$  and  $\tau_\gamma$  is the horizontal vector field (5.13). If  $\gamma$  is a Hamiltonian connection such that  $\tau_\gamma \lrcorner \Omega = dH$ , we have

$$\tau_{\rho(\gamma)} \lrcorner \Omega = (\rho^{-1})^*(\tau_\gamma \lrcorner \Omega) = d((\rho^{-1})^*H).$$

●

A glance at the relation (5.22) shows that, for each locally Hamiltonian connection  $\gamma$ , the horizontal Hamiltonian vector field  $\tau_\gamma$  is the generator of a local 1-parameter group  $G_\gamma$  of canonical morphisms of  $\Pi \rightarrow X$ . It leads to the following assertion.

PROPOSITION 5.8. Let  $X = \mathbf{R}$  and  $\gamma$  be a complete locally Hamiltonian connection on  $\Pi \rightarrow \mathbf{R}$ . There exist canonical coordinate transformations which bring  $\gamma$  into zero.  $\square$

**Proof.** In virtue of Proposition 2.10, there exists a trivialization of the bundle  $\Pi \rightarrow \mathbf{R}$  such that  $\gamma^i = 0$ ,  $\gamma_i = 0$  relative to coordinates which are constant along the integral curves of  $\tau_\gamma$ . Since  $G_\gamma$  is a group of canonical transformations, we deduce that the above-mentioned coordinates are canonical. ●

} From physical viewpoint, the above coordinates are the initial values of the canonical variables. }

COROLLARY 5.9. The evolution equations (5.9) associated with a Hamiltonian connection can be locally brought into the equilibrium equations

$$y_t^i = 0, \quad p_{it} = 0$$

by canonical transformations.  $\square$

**Example 5.4.** Let us consider 1-dimensional motion with constant acceleration  $a$  with respect to a reference frame whose coordinates are  $(t, y)$ . The corresponding Hamiltonian and the Hamiltonian connection read

$$\begin{aligned} \mathcal{H} &= \frac{p^2}{2} - ay, \\ \gamma^y &= p, \quad \gamma_p = a. \end{aligned} \tag{5.32}$$



This is a complete connection. The canonical transformation

$$y' = y - pt + \frac{at^2}{2}, \quad p' = p - at$$

brings the connection (5.32) into zero. ●

**Example 5.5.** Let us consider the 1-dimensional oscillator with respect to the same frame. The Hamiltonian and Hamiltonian connection of the oscillator read

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}(p^2 + y^2), \\ \gamma^y &= p, \quad \gamma_p = -y. \end{aligned} \tag{5.33}$$

This is a complete connection. The canonical transformation

$$y' = y \cos t - p \sin t, \quad p' = p \cos t + y \sin t$$

brings the connection (5.33) into zero. ●

**Example 5.6.** Let us consider 1-dimensional motion in a viscous medium with respect to the reference frame in the previous Examples. It is described by the first order differential evolution equation

$$y_t = \gamma^y, \quad p_t = \gamma_p, \tag{5.34}$$

where

$$\gamma^y = p, \quad \gamma_p = -p \tag{5.35}$$

is a connection on  $\Pi$ . It is a complete connection, but not locally Hamiltonian. The coordinate transformation

$$y' = y + p(1 - e^t), \quad p' = pe^t \tag{5.36}$$

brings the connection (5.35) into zero so that the equations (5.34) come to the equilibrium equations

$$y'_t = 0, \quad p'_t = 0.$$

However, (5.36) is not a canonical transformation. ●

It should be emphasized that, in general, the canonical transformations introduced above do not preserve the splitting (5.7). Consequently, they do not send a Hamiltonian form into a Hamiltonian form and do not maintain the form of the Hamilton equations (5.8) in general.

At the same time, Proposition 5.7 shows that canonical morphism send general Hamiltonian forms to general Hamiltonian forms.

**PROPOSITION 5.10.** Let  $\gamma$  be a Hamiltonian connection on  $\Pi \rightarrow X$  and  $H$  the corresponding general Hamiltonian form (see Definition 4.2). In virtue of Proposition 5.1, we have  $\tau_\gamma \rfloor \Omega = dH$ . Let  $H'$  be another general Hamiltonian form. Then,  $\sigma = H' - H$  is a 1-form on  $\Pi$  such that

$$\begin{aligned} d(\sigma \wedge dt) &= 0, \\ \partial_j \sigma_i - \partial_i \sigma_j &= 0, \quad \partial^j \sigma^i - \partial^i \sigma^j = 0, \quad \partial^j \sigma_i - \partial_i \sigma^j = 0. \end{aligned} \tag{5.37}$$

□

It follows that, if  $\rho$  is a canonical morphism and  $H$  is a Hamiltonian form, then

$$\rho^* H = H - \sigma = p_i dy^i - (\mathcal{H} + \sigma_t) dt - \sigma_i dy^i - \sigma^i dp_i,$$

where  $\sigma = \sigma_t dt + \sigma_i dy^i + \sigma^i dp_i$  is a 1-form on  $\Pi$  which satisfies (5.37). Accordingly, the Hamilton equations (5.10a) – (5.10b) are brought into the form

$$\begin{aligned} y_t^i &= \partial^i (\mathcal{H} + \sigma_t) + \partial_t \sigma^i, \\ p_{it} &= -\partial^i (\mathcal{H} + \sigma_t) - \partial_t \sigma^i. \end{aligned}$$

**Remark 5.7.** Every general Hamiltonian form is Hamiltonian locally. Every canonical morphism  $\rho$  transforms a Hamiltonian form to a Hamiltonian form locally since the condition (5.37) implies that  $\sigma = f dt + dS$  locally, where  $f$  and  $S$  are local functions on  $\Pi$ . ●

Canonical transformations keep the Hamilton equations if

$$\rho^* H = H - dS, \tag{5.38}$$

where  $S$  is a function on  $\Pi$  called the *generating function*. In this case, one says that  $\rho^* H$  and  $H$  describe the same mechanical system. The relation (5.38) can be written as the Pfaffain equation on the graph  $\Pi_\rho \subset \Pi \times \Pi$  of the canonical morphism  $\rho$ .

In particular, assume that the graph  $\Pi_\rho$  is coordinatized by  $(t, y^i, y'^i)$ . The equality (5.38) takes the coordinate form

$$p'_i dy'^i - p_i dy^i + (\mathcal{H} - \mathcal{H}') dt = -dS(t, y^i, y'^i).$$

It leads to the familiar relations

$$p_i = \frac{\partial S}{\partial y^i}, \quad p'_i = -\frac{\partial S}{\partial y'^i}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial S}{\partial t}.$$

**Example 5.8.** The holonomic coordinate transformations (5.3) admit locally the generating function  $S(t, y'^j, p_i) = y^i(t, y'^j) p_i$ . ●

## 5.5 Vertical extension of the Hamiltonian formalism

We now turn to the vertical extension of the time-dependent Hamiltonian formalism (see Section 4.7) in order to make Hamiltonian forms and Hamilton equations invariant under canonical transformations. In case of symplectic mechanics, the similar extension of the symplectic geometry from  $T^*M$  to  $TT^*M$  has been considered in [1, 51].

Given a bundle  $Y \rightarrow X$ , let us consider the Legendre manifold  $\Pi_{VY}$  corresponding to the bundle  $VY \rightarrow X$ . It is isomorphic to the vertical tangent bundle  $V\Pi = VV^*Y$  of  $\Pi \rightarrow X$  (see Proposition 4.8). We call  $\Pi_{VY}$  the *vertical phase space* and provide it with the coordinates  $(x^\lambda, y^i, p_i, \dot{y}^i, \dot{p}_i)$  of  $V\Pi$  (recall the notations (2.6) and (4.29)).

The canonical form  $\mathbf{\Lambda}$  (5.4) on  $V\Pi$  is the  $n = 1$  reduction

$$\boxed{\mathbf{\Lambda}_V = [d\dot{p}_i \wedge dy^i + dp_i \wedge d\dot{y}^i] \wedge dt \otimes \partial_t} \quad (5.39)$$

of the form (4.30). The canonical form  $\mathbf{\Omega}$  (5.20) on  $V\Pi$  reads

$$\boxed{\mathbf{\Omega}_V = [d\dot{p}_i \wedge dy^i + dp_i \wedge d\dot{y}^i] \wedge dt.} \quad (5.40)$$

With the canonical form (5.40), the vertical phase space  $V\Pi$  can be equipped with the canonical Poisson structure (5.18) given by the bracket

$$\boxed{\{f, g\}_{VV} = \dot{\partial}^i f \partial_i g + \partial^i f \dot{\partial}_i g - \partial^i g \dot{\partial}_i f - \dot{\partial}^i g \partial_i f.} \quad (5.41)$$

The notions of Hamiltonian connection, Hamiltonian vector field, horizontal Hamiltonian vector field and Hamiltonian form on  $V\Pi$  are the straightforward generalization of those in Section 5.1.

Every Hamiltonian form  $H_V$  on  $V\Pi$  admits the splitting

$$\boxed{H_V = \dot{p}_i dy^i - \dot{y}^i dp_i - \mathcal{H}_V, \quad \mathcal{H}_V = (\dot{p}_i \tilde{\gamma}^i - \dot{y}^i \tilde{\gamma}_i + \widetilde{\mathcal{H}}_V) dt,} \quad (5.42)$$

where  $\gamma$  is a connection on  $\Pi \rightarrow X$ . The corresponding Hamilton equations (5.8) takes the form

$$\gamma^i = \dot{\partial}^i \mathcal{H}_V, \quad (5.43a)$$

$$\gamma_i = -\dot{\partial}_i \mathcal{H}_V, \quad (5.43b)$$

$$\dot{\gamma}^i = \partial^i \mathcal{H}_V, \quad (5.43c)$$

$$\dot{\gamma}_i = -\partial_i \mathcal{H}_V. \quad (5.43d)$$

In particular, the vertical lift  $V\tilde{\gamma}$  (2.31) of a Hamiltonian connection  $\tilde{\gamma}$  associated with a Hamiltonian  $\mathcal{H}$  on  $\Pi$  is the Hamiltonian connection associated with the Hamiltonian

$$\mathcal{H}_V = \partial_V \mathcal{H} = (\dot{y}^i \partial_i + \dot{p}_i \partial^i) \mathcal{H}$$

on VII (see Proposition 4.9). In this case, the Hamilton equations (5.43a) and (5.43b) are exactly the Hamilton equations (5.8) for the Hamiltonian connection  $\tilde{\gamma}$ .

Let us consider the canonical coordinate transformations of the Legendre bundle  $\Pi \rightarrow X$  and the induced (holonomic) coordinate transformations

$$\dot{p}'_i = \partial_V p'_i, \quad \dot{y}'^i = \partial_V y'^i \quad (5.44)$$

of VII. It is readily observed that they are also canonical transformations for the canonical forms (5.39), (5.40). They are linear in the coordinates  $\dot{p}_i, \dot{y}^i$  and, obviously, do not exhaust all canonical transformations of VII. These transformations maintain the Poisson bracket (5.41). The splitting (5.42) of a Hamiltonian  $\mathcal{H}_V$  and the Hamilton equations (5.43a) – (5.43d) also are invariant under the canonical transformations (5.44). We have

$$H_V = \dot{p}_i dy^i - \dot{y}^i dp_i - \mathcal{H}_V = \dot{p}'_i dy'^i - \dot{y}'^i dp'_i - \mathcal{H}'_V,$$

where

$$\mathcal{H}'_V = \mathcal{H}_V - (\partial_V p'_i \partial_t y'^i - \partial_V y'^i \partial_t p'_i). \quad (5.45)$$

At the same time, if  $\mathcal{H}_V = \partial_V \mathcal{H}$ , where  $\mathcal{H}$  is a Hamiltonian on  $\Pi$ , the Hamiltonian  $\mathcal{H}'_V$  (5.45) fails to represent the derivative  $\partial_V$  of some Hamiltonian on  $\Pi$  in general.

**PROPOSITION 5.11.** Every connection  $\tilde{\gamma}$  on the Legendre bundle  $\Pi$  gives rise to the Hamiltonian connection on VII.  $\square$

**Proof.** Let us consider the Hamiltonian form

$$H_V = \dot{p}_i (dy^i - \tilde{\gamma}^i dt) - \dot{y}^i (dp_i - \tilde{\gamma}_i dt) = \dot{p}_i dy^i - \dot{y}^i dp_i - (\dot{p}_i \tilde{\gamma}^i - \dot{y}^i \tilde{\gamma}_i) dt$$

The corresponding Hamiltonian connection on VII is given by the Hamilton equations (5.43a) – (5.43d) which take the form

$$\dot{\gamma}^i = \tilde{\gamma}^i, \quad \dot{\gamma}_i = \tilde{\gamma}_i, \quad \dot{\gamma}^i = \dot{p}_j \partial^i \tilde{\gamma}^j - \dot{y}^j \partial^i \tilde{\gamma}_j, \quad \dot{\gamma}_i = -\dot{p}_j \partial_i \tilde{\gamma}^j + \dot{y}^j \partial_i \tilde{\gamma}_j. \quad (5.46)$$

In particular, if  $\tilde{\gamma}$  is a Hamiltonian connection on  $\Pi$ , the Hamiltonian connection (5.46) coincides with the vertical connection  $V\gamma$  (2.31).  $\bullet$

It follows that every first order evolution equations (5.9) on the Legendra bundle  $\Pi$  can be written as a part (5.43a), (5.43b) of the Hamilton equations on VII.

**Example 5.9.** The 1-dimensional motion in a viscous medium in Example 5.6 is described by the Hamiltonian  $\mathcal{H}_V = p(\dot{p} + \dot{y})$  on VII.  $\bullet$

## 5.6 Reference frames

The form of the Hamiltonian evolution equation (5.24) is maintained under canonical transformations of  $\Pi$  when

$$\widetilde{\mathcal{H}}'(t, y'^j, p'_j) = \widetilde{\mathcal{H}}(t, y^j, p_j), \quad (\partial_t + \gamma_0^i \partial_i + \gamma_{0i} \partial^i) f = (\partial_t + \gamma_0'^i \partial'_i + \gamma_{0i}' \partial'^i) f'.$$

In virtue of Corollary 5.9, we can make locally the Hamiltonian connection  $\gamma_0$  equal to zero by canonical coordinate transformations and can bring the Hamiltonian evolution equation (5.24) into the familiar Poisson bracket form

$$d_{Ht} f = \partial_t f + \{\widetilde{\mathcal{H}}, f\}_V. \quad (5.47)$$

In virtue of Proposition 5.8, we can get this form of the Hamiltonian evolution equation with respect to the global trivialization of  $\Pi$  if  $X = \mathbf{R}$  and the Hamiltonian connection  $\gamma_0$  in the splitting (5.23) is complete.

In particular, let  $\Gamma$  be a complete connection on the bundle  $Y \rightarrow \mathbf{R}$  associated with some trivialization (5.1) of  $Y$ . Then, the connection  $\tilde{\Gamma} = V^* \Gamma$  (5.5) is a complete Hamiltonian connection on  $\Pi$  associated with the corresponding trivialization (5.2) of  $\Pi$ . It follows that we can utilize the covector lift  $\gamma_0 = \tilde{\Gamma}$  of a complete connection  $\Gamma$  on  $Y$  in order to bring the Hamiltonian evolution equation (5.24) into the Poisson bracket form (5.47).

**DEFINITION 5.12.** We say that a complete connection  $\Gamma$  on  $Y \rightarrow X$  describes a *reference frame* in time-dependent mechanics.  $\square$

Indeed, the difference of  $\Gamma' - \Gamma = u dt$  defines a vertical vector field  $u$  on  $Y$  which characterizes the relative velocities between reference frames  $\Gamma'$  and  $\Gamma$ . Accordingly, one can think of

$$\dot{y}^i \circ D_\Gamma = y_t^i - \Gamma^i$$

as being the relative velocities of a mechanical system with respect to the reference frame  $\Gamma$ . By Definition 5.12, there is the 1:1 correspondence between reference frames and trivializations of  $Y \rightarrow \mathbf{R}$ .

One can say that a reference frame provides a splitting between the time and the other coordinates of a mechanical system.  $\}$

**Remark 5.10.** Every connection  $\Gamma$  on  $Y \rightarrow X$  defines a local reference frame. In virtue of Proposition 4.6, every Hamiltonian form  $H$  on  $\Pi$  defines the connection  $\Gamma_H$  on  $Y \rightarrow X$

and, consequently, the corresponding local reference frame. We call  $\Gamma_H$  the (local) *proper reference frame*. With respect to this reference frame, the Hamilton equations (5.10a) takes the form

$$y_t^i - \Gamma_H^i = \partial^i \widetilde{\mathcal{H}}. \quad (5.48)$$

One can think of these equations as being the relations between the canonical momenta  $p_i$  and the velocities  $y_t^i - \Gamma_H^i$  relative to the proper reference frame. In accordance with the definition of  $\Gamma_H$ , this relation implies that the null momenta corresponds to the null velocities (5.48). ●

## 6 Lagrangian mechanics

We aim to investigate the relations between Lagrangian and Hamiltonian formulations of time-dependent mechanics. From the mathematical point of view, these formulations are not equivalent in case of degenerate Lagrangians. From physical viewpoint, velocities are physical observables in classical mechanics, whereas momenta are physical observables in quantum mechanics.

Given a bundle  $Y \rightarrow X$  over a 1-dimensional base  $X$ , the Lagrangian mechanics of sections of  $Y \rightarrow X$  is formulated on the configuration space  $J^1Y$  coordinatized by  $(t, y^i, y_t^i)$ . A Lagrangian on  $J^1Y$  reads  $L = \mathcal{L}dt$ . Also recall the notation

$$\pi_i = \partial_i^t \mathcal{L}, \quad \pi_{ij} = \partial_j^t \partial_i^t \mathcal{L}, \quad \tilde{\pi} = \mathcal{L} - \pi_i y_t^i.$$

### 6.1 Poisson structure

In contrast with Hamiltonian mechanics, the configuration space  $J^1Y$  of Lagrangian mechanics possesses no canonical Poisson structure.

Given a Lagrangian  $L$  on  $J^1Y$ , the pullbacks of the canonical forms on  $V^*Y$  and  $T^*Y$  are defined by the Legendre morphism  $\hat{L} : J^1Y \rightarrow V^*Y$  (4.9) and the morphism  $\hat{\Xi}_L : J^1Y \rightarrow T^*Y$  (4.12) on  $J^1Y$ .

Let  $\Omega$  be the canonical 3-form (5.20) on  $V^*Y$ . Its pullback by the Legendre morphism  $\hat{L}$  reads

$$\Omega_L = \hat{L}^* \Omega = d\pi_i \wedge dy^i \wedge dt = (\pi_{ij} dy_t^j \wedge dy^i + \partial_j \pi_i dy^j \wedge dy^i) \wedge dt.$$

Using  $\Omega_L$ , every vertical vector field  $\vartheta = \vartheta^i \partial_i + \dot{\vartheta}^i \partial_i^t$  on  $J^1Y \rightarrow X$  is sent to the 2-form

$$\vartheta \rfloor \Omega_L = \{[\dot{\vartheta}^j \pi_{ji} + \vartheta^j (\partial_j \pi_i - \partial_i \pi_j)] dy^i - \vartheta^i \pi_{ji} dy_t^j\} \wedge dt.$$

If the Lagrangian  $L$  is regular ( $\det \pi_{ij} \neq 0$ ), the above map is a bijection. Indeed, given any 2-form  $\phi = (\phi_i dy^i + \dot{\phi}_i dy_t^i) \wedge dt$ , the algebraic equations

$$\dot{\vartheta}^j \pi_{ji} + \vartheta^j (\partial_j \pi_i - \partial_i \pi_j) = \phi_i, \quad -\vartheta^i \pi_{ji} = \dot{\phi}_j$$

have the unique solution

$$\vartheta^i = -(\pi^{-1})^{ij} \dot{\phi}_j, \quad \dot{\vartheta}^j = (\pi^{-1})^{ji} [\phi_i + (\pi^{-1})^{kn} \dot{\phi}_n (\partial_k \pi_i - \partial_i \pi_k)].$$

In particular, every function  $f$  on  $J^1Y$  defines a vertical vector field

$$\vartheta_f = -(\pi^{-1})^{ij} \partial_j^t f \partial_i + (\pi^{-1})^{ji} [\partial_i f + (\pi^{-1})^{kn} \partial_n^t f (\partial_k \pi_i - \partial_i \pi_k)] \partial_j^t. \quad (6.1)$$

Following the relation (5.21), one can introduce the Poisson structure on the space  $S(J^1Y)$  of functions on  $J^1Y$ . It is given by the bracket

$$\begin{aligned} \vartheta_g \rfloor \vartheta_f \rfloor \Omega_L &= \{f, g\}_L dt, \\ \{f, g\}_L &= [(\pi^{-1})^{ij} + (\partial_n \pi_k - \partial_k \pi_n)(\pi^{-1})^{ki}(\pi^{-1})^{nj}](\partial_i^t f \partial_j g - \partial_i^t g \partial_j f) + \\ &\quad (\partial_n \pi_k - \partial_k \pi_n)(\pi^{-1})^{ki}(\pi^{-1})^{nj} \partial_i^t f \partial_j^t g. \end{aligned} \quad (6.2)$$

The vertical vector field  $\vartheta_f$  (6.1) is the Hamiltonian vector field of the function  $f$  with respect to this Poisson structure.

In particular, if the Lagrangian  $L$  is hyperregular, that is, the Legendre morphism  $\hat{L}$  is diffeomorphism, the Poisson structure (6.2) is obviously isomorphic to the Poisson structure (5.18) on the phase space  $\Pi = V^*Y$ .

The Poisson structure (6.2) defines the corresponding symplectic foliation on  $J^1Y$  which consists with the fibration  $J^1Y \rightarrow X$ . The symplectic form on the leaf  $J_x^1Y$  of this foliation is  $\Omega_x = d\pi_i \wedge dy^i$  [53].

## 6.2 Spray-like equations

In the framework of Hamiltonian mechanics above, we have shown that the choice of a trivialization  $Y \simeq X \times M$  corresponds to the choice of a certain reference frame. We here illustrate this fact in case of evolution equations on the configuration space. We consider second order evolution equations which are not necessarily of Lagrangian type.

Let us recall the notion of spray in autonomous mechanics. Let  $M$  be a manifold coordinatized by  $(y^i)$  and

$$K = dy^i \otimes (\partial_i - K^k_{ji} \dot{y}^j \partial_k)$$

a linear connection (2.30) on  $TM$ . It yields the vector field

$$\dot{y}^i \partial_i \rfloor K(y, \dot{y}) = \dot{y}^i (\partial_i - K^k_{ji} \dot{y}^j \partial_k) \quad (6.3)$$

on  $TM$  which is called the *geodesic spray*. The equations

$$\frac{dy^i}{dt} = \dot{y}^i, \quad \frac{d\dot{y}^i}{dt} = -K^i_{ji} \dot{y}^j \dot{y}^i$$

for integral curves of the spray (6.3) are second order differential equations whose solutions are geodesics of the connection  $K$ .

We aim to discover similar spray-like equations in time-dependent mechanics.



Given a bundle  $Y \rightarrow X$  over a 1-dimensional base, let

$$A = dt \otimes (\partial_t + A^i \partial_i^t) + dy^j \otimes (\partial_j + A_j^i \partial_i^t) \quad (6.4)$$

be a connection on the jet bundle  $J^1 Y \rightarrow Y$ . It has the transformation law

$$\begin{aligned} A'^i_k &= \left( \frac{\partial y'^i}{\partial y^j} A^j_n + \frac{\partial y'^i}{\partial y^n} \right) \frac{\partial y^n}{\partial y'^k}, \\ A'^i &= \left( \frac{\partial y'^i}{\partial y^j} A^j + \frac{\partial y'^i}{\partial t} \right) + \left( \frac{\partial y'^i}{\partial y^j} A^j_k + \frac{\partial y'^i}{\partial y^k} \right) \frac{\partial y^k}{\partial t} = \frac{\partial y'^i}{\partial y^j} A^j + \frac{\partial y'^i}{\partial t} - A'^i_k \frac{\partial y'^k}{\partial t}. \end{aligned} \quad (6.5)$$

**DEFINITION 6.1.** Given a connection  $A$  (6.4) on  $J^1 Y \rightarrow Y$ , by a *second order evolution equation* on  $Y$  is meant the restriction of the kernel  $\text{Ker } \widetilde{D} \subset J^1 J^1 Y$  of the vertical covariant differential  $\widetilde{D}$  (2.41) to  $J^2 Y$ . This is given by the coordinate relation

$$y_{tt}^i = A^i + A_{tj}^i y^j. \quad (6.6)$$

□

A glance at the expression (6.6) shows that different connections (6.4) can lead to the same evolution equation.

**Remark 6.1.** Every connection (6.4) on  $J^1 Y \rightarrow Y$  generates the connection

$$\gamma = dt \otimes (\partial_t + y_t^i \partial_i + (A^i + A_j^i y_t^j) \partial_i^t) \quad (6.7)$$

on  $J^1 Y \rightarrow X$ . The horizontal lift of the vector field  $\partial_t$  on  $X$  onto  $J^1 Y$  by means of the connection (6.7) reads

$$\partial_t + y_t^i \partial_i + (A^i + A_j^i y_t^j) \partial_i^t.$$

The integral curves of this vector field are the generalized solutions of the evolution equations (6.6). Conversely, second order evolution equation can be often defined as the equation

$$d_t y^i = y_t^i, \quad d_t y_t^i = \xi^i,$$

for an integral curve of a vector field

$$\partial_t + y_t^i \partial_i + \xi^i \partial_i^t$$

on  $J^1 Y$ . Every such a vector field defines a connection

$$A_j^i = \frac{1}{2} \frac{\partial \xi^i}{\partial y_t^j}, \quad A^i = \xi^i - A_j^i$$

on  $J^1Y \rightarrow Y$  [17] which lead to the same evolution equation in accordance with Definition 6.1. •

In particular, let  $Y \rightarrow X$  be a trivializable bundle and  $Y \simeq X \times M$  its trivialization with the coordinates  $(t, y^i)$  whose transition functions are independent on  $t$ . We have the corresponding trivialization  $J^1Y \simeq X \times TM$  with the coordinates  $(t, y^i, \dot{y}^i)$ , where  $\dot{y}^i$  are holonomic coordinates of  $TM$ . With respect to these coordinates, the transformation law (6.5) of the connection (6.4) reads

$$A'^i = \frac{\partial y'^i}{\partial y^j} A^j \quad A'^i_k = \left( \frac{\partial y'^i}{\partial y^j} A^j_n + \frac{\partial \dot{y}'^i}{\partial y^n} \right) \frac{\partial y^n}{\partial y'^k}. \quad (6.8)$$

A glance at the expression (6.8) shows that, given a trivialization of  $Y \rightarrow X$ , a connection on  $J^1Y \rightarrow Y$  defines a vertical time-dependent vector field  $A^i_t$  on  $TM$  and a time-dependent connection on  $TM \rightarrow M$ . The converse procedure enables us to discover a spray-like equation on  $Y$ .

Let  $K^i_k(y^j, \dot{y}^j)$  be a connection (e.g., a linear connection) on  $TM \rightarrow M$ . Given the above-mentioned trivialization of  $Y$ , the connection  $K$  defines the connection  $A$  on  $J^1Y \rightarrow Y$  by the coordinate relations

$$A^i = 0, \quad A^i_k = K^i_k. \quad (6.9)$$

Owing to the transformation law (6.5), we can write the connection (6.9) with respect to arbitrary bundle coordinates  $(t, y^i)$ . It reads

$$A^i_k = \left[ \frac{\partial y^i}{\partial y^j} K^j_n(y^j(y^i), \dot{y}^j(y^i, y^i_t)) + \frac{\partial^2 y^i}{\partial y^n \partial y^j} \dot{y}^j + \frac{\partial \Gamma^i}{\partial y^n} \right] \partial_k y^n, \quad (6.10)$$

$$A^i = \partial_t \Gamma^i + \partial_j \Gamma^i y^j_t - A^i_k \Gamma^k,$$

where  $\Gamma^i = \partial_t y^i$  is the connection on  $Y \rightarrow X$  which corresponds to the initial trivialization of  $Y$ , that is,  $\Gamma = 0$  relative to the coordinates  $(t, y^i)$ .

**Remark 6.2.** Given the connection  $A$  (6.9) on  $J^1Y \rightarrow Y$  and the above-mentioned connection  $\Gamma$  on  $Y \rightarrow X$ , the corresponding composite connection (2.40) consists with the jet lift  $J\Gamma$  (2.33) of  $\Gamma$  onto  $J^1Y \rightarrow Y$ . We have

$$J\Gamma = dt \otimes (\partial_t + \Gamma^i \partial_i + d_t \Gamma^i \partial^t_i). \quad (6.11)$$

•

The evolution equation (6.6) with respect to the connection (6.10) reads

$$y^i_{tt} = \partial_t \Gamma^i + y^j_t \partial_j \Gamma^i + A^i_k (y^k_t - \Gamma^k). \quad (6.12)$$

Given a reference frame  $Y \simeq X \times M$  with coordinates  $(t, y^i)$ , let  $K_j^i = 0$ . In accordance with (6.9), this choice leads to the free motion equation

$$\ddot{y}^i = 0. \quad (6.13)$$

With respect to arbitrary bundle coordinates  $(t, y^i)$ , this equation reads

$$y_{tt}^i = \partial_t \Gamma^i - \Gamma^j \partial_j \Gamma^i + y_t^k (2\partial_k \Gamma^i + \frac{\partial y^i}{\partial y^j} \frac{\partial y^j}{\partial y^m \partial y^k} \Gamma^m) - \frac{\partial y^i}{\partial y^j} \frac{\partial y^j}{\partial y^m \partial y^k} y_t^k y_t^m. \quad (6.14)$$

One can treat the right side of this equation as the general expression for inertial forces.

Such kind of terms in spray-like evolution equations can be always excluded by the choice of a reference frame.

**Remark 6.3.** The equations (6.13) are obviously of Lagrangian type, but not the equations (6.14). At the same time, the equations (6.14) are equivalent to the Euler–Lagrange equations of the Lagrangian

$$\mathcal{L} = \frac{1}{2} \delta_{ab} \frac{\partial y^a}{\partial y'^i} \frac{\partial y^b}{\partial y'^j} (y'^i_t - \Gamma^i) (y'^j_t - \Gamma^j).$$

●

**Example 6.4.** Let us consider a free point on a plain. Let the splitting  $Y = \mathbf{R} \times \mathbf{R}^2$  with coordinates  $(t, y^1, y^2)$  corresponds to an inertial reference frame. Let the connection  $K$  on the bundle  $T\mathbf{R}^2$  be equal to zero. Let us consider the rotating reference frame with the coordinates

$$y^1 = y^1 \cos wt - y^2 \sin wt, \quad y^2 = y^2 \cos wt + y^1 \sin wt.$$

With respect to this reference frame, the equation (6.14) reads

$$y_{tt}^i = \partial_t \Gamma^i + 2y_t^j \partial_j \Gamma^i - \Gamma^j \partial_j \Gamma^i, \quad (6.15)$$

where

$$\Gamma^1 = \partial_t y^1 = -wy^2, \quad \Gamma^2 = \partial_t y^2 = wy^1. \quad (6.16)$$

Substituting (6.16) into (6.15), we find

$$y_{tt}^1 = w^2 y^1 - 2wy^2, \quad y_{tt}^2 = w^2 y^2 + 2wy^1.$$

•

This Example shows that, on physical level, we can treat  $y_t^i$  in the evolution equation (6.12) as the velocities relative to the (local) reference frame given by the connection on  $Y \rightarrow X$  which vanishes with respect to these coordinates (see Section 5.6).

At the same time, the evolution equation (6.12) is brought into the spray-like form

$$\begin{aligned} d_t \dot{y}^i &= K_k^i \dot{y}^k + \partial_k \Gamma^i \dot{y}^k, \\ K_k^i &= \frac{\partial y^i}{\partial y^j} \frac{\partial y^n}{\partial y^k} K_n^j(y^j(y^i), \dot{y}^j(y^i, \dot{y}^i)) - \frac{\partial y^i}{\partial y^j} \frac{\partial y^j}{\partial y^m \partial y^k} \dot{y}^m, \\ \dot{y}^i &= y_t^i - \Gamma^i, \quad \dot{y}^i = \frac{\partial y^i}{\partial y^k} \dot{y}^k, \end{aligned}$$

where  $\dot{y}^i \partial_i$  can be treated the relative velocities with respect to the initial reference frame  $\Gamma$  which are written with respect to the coordinates  $y^i$ .

In particular, if  $K_k^i = -K_{mk}^i \dot{y}^m$  is a linear connection on  $TM$ , we have

$$\begin{aligned} d_t \dot{y}^i &= -K_{mk}^i \dot{y}^m \dot{y}^k + \partial_k \Gamma^i \dot{y}^k, \\ K_{mk}^i &= \frac{\partial y^i}{\partial y^j} \frac{\partial y^n}{\partial y^k} \frac{\partial y^l}{\partial y^m} K_{lk}^j - \frac{\partial y^i}{\partial y^j} \frac{\partial y^j}{\partial y^m \partial y^k} \dot{y}^m. \end{aligned}$$

### 6.3 Hamiltonian and Lagrangian formalisms

According to Section 4.6, we establish the relations between Lagrangian and Hamiltonian formalisms for time-dependent mechanical systems.

Let  $Y \rightarrow X$  be a bundle over a 1-dimensional base,  $\Pi = V^*Y$  the phase space and  $J^1Y$  the configuration space.

**DEFINITION 6.2.** A Hamiltonian  $\mathcal{H}$  on  $\Pi$  is said to be associated with a Lagrangian  $L$  on  $J^1Y$  if it obeys the conditions

$$\boxed{\widehat{L} \circ \widehat{H}|_Q = \text{Id}_Q, \quad p_i = \partial_i^t \mathcal{L}(t, y^j, \partial^j \mathcal{H}(p)), \quad p \in Q = \widehat{L}(J^1Y)} \quad (6.17a)$$

$$\boxed{H_{\widehat{H}} - H = L \circ \widehat{H}, \quad \mathcal{L}(t, y^j, \partial^j \mathcal{H}) \equiv p_i \partial^i \mathcal{H} - \mathcal{H}.} \quad (6.17b)$$

□

Also the relation

$$\boxed{\partial_i \mathcal{H} + \partial_i \mathcal{L} = 0} \quad (6.18)$$

takes place on  $Q$ .

If a Lagrangian  $L$  is hyperregular, there exists a unique Hamiltonian associated with  $L$ . If a Lagrangian  $L$  is degenerate, different Hamiltonians or no Hamiltonian at all may be associated with  $L$  in general.

**PROPOSITION 6.3.** Let a Lagrangian  $L$  be *almost regular*, that is, the Lagrangian constraint space  $Q$  is an imbedded submanifold of  $\Pi$  and the Legendre morphism  $\hat{L} : J^1Y \rightarrow Q$  is a submersion. Then, each point of  $Q$  has an open neighborhood on which there exists a local Hamiltonian form associated with  $L$  [48, 56].  $\square$

**Example 6.5.** Let  $Y$  be the bundle  $\mathbf{R}^2 \rightarrow \mathbf{R}$  coordinatized by  $(t, y)$ . Consider the Lagrangian  $\mathcal{L} = \exp y_t$ . It is regular and semiregular, but not hyperregular. The corresponding Legendre morphism reads  $p \circ \hat{L} = \exp y_t$ . The image  $Q$  of the configuration space under this morphism is given by the coordinate relation  $p > 0$ . It is an open subbundle of the Legendre bundle. On  $Q$ , we have the associated Hamiltonian

$$\mathcal{H} = p(\ln p - 1)$$

which however can not be extended to  $\Pi$ .  $\bullet$

All Hamiltonian forms associated with a semiregular Lagrangian  $L$  coincide with each other on the Lagrangian constraint space  $Q$ , and the Poincaré–Cartan form  $\Xi_L$  (4.3) is the pullback

$$\boxed{\pi_i y_t^i - \mathcal{L} \equiv \mathcal{H}(t, y^i, \pi_i),}$$

of such a Hamiltonian form  $H$  by the Legendre morphism  $\hat{L}$ .

**Example 6.6.** Let  $Y$  be the bundle  $\mathbf{R}^3 \rightarrow \mathbf{R}$  coordinatized by  $(t, y^1, y^2)$ . Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(y_t^1)^2. \tag{6.19}$$

It is semiregular. The associated Legendre morphism reads

$$p_1 \circ \hat{L} = y_t^1, \quad p_2 \circ \hat{L} = 0.$$

The corresponding constraint space  $Q$  consists of points with the coordinate  $p_2 = 0$ . The Hamiltonians associated with the Lagrangian (6.19) are given by the expression

$$\mathcal{H} = \frac{1}{2}(p_1)^2 + c(t, y)p_2, \tag{6.20}$$

where  $c$  is arbitrary function on  $Y$ . They coincide with each other on  $Q$ .  $\bullet$

**COROLLARY 6.4.** In accordance with Proposition 4.7, if  $\mathcal{H}$  is a Hamiltonian associated with a semiregular Lagrangian  $L$ , every solution of the corresponding Hamilton equations which lives on the Lagrangian constraint space  $Q$  yields a solution of the Euler–Lagrange equations for  $L$ . At the same time, to exhaust all solutions of the Euler–Lagrange equations, one must consider a complete family (if it exists) of Hamiltonians associated with  $L$ .  $\square$

For instance, the Hamiltonians (6.20) associated with the Lagrangian (6.19) constitute a complete family.

**PROPOSITION 6.5.** If a Lagrangian  $L$  is semiregular and almost regular, then every point of  $Q$  has a neighborhood on which there exists a complete family of local Hamiltonians associated with  $L$  [48, 56].  $\square$

The following example shows that a complete family of associated Hamiltonians may exist even if a Lagrangian is neither semiregular nor almost regular.

**Example 6.7.** Let  $Y$  be the bundle  $\mathbf{R}^2 \rightarrow \mathbf{R}$ . Let us consider the Lagrangian

$$\mathcal{L} = \frac{1}{3}(y_t)^3.$$

The associated Legendre morphism reads

$$p \circ \hat{L} = (y_t)^2. \tag{6.21}$$

The corresponding constraint space  $Q$  is given by the coordinate relation  $p \geq 0$ . It is not even a submanifold of  $\Pi$ . There exist two associated Hamiltonians

$$\mathcal{H}_+ = \frac{2}{3}p^{\frac{3}{2}}, \quad \mathcal{H}_- = -\frac{2}{3}p^{\frac{3}{2}}$$

which are defined only on the constraint space  $Q$ . They correspond to different solutions

$$y_t = \sqrt{p}, \quad y_t = -\sqrt{p}$$

of the equation (6.21) and constitute a complete family.  $\bullet$

## 6.4 Quadratic Lagrangians and Hamiltonians

As an important illustration of Proposition 6.5, let us describe the complete families of Hamiltonians associated with almost regular quadratic Lagrangians.

**Remark 6.8.** Since Hamiltonians in time-dependent mechanics are not functions on a phase space, we can not apply to them the well-known analysis of the normal forms [7] (e.g. quadratic Hamiltonians in symplectic mechanics [2]).  $\bullet$

Let us consider a quadratic Lagrangian

$$\mathcal{L} = \frac{1}{2}a_{ij}(y)y_t^i y_t^j + b_i(y)y_t^i + c(y) \quad (6.22)$$

on  $J^1Y$ , where  $a$ ,  $b$  and  $c$  are local functions on  $Y$  with the corresponding transformation laws. The associated Legendre morphism reads

$$p_i \circ \widehat{L} = a_{ij}y_t^j + b_i. \quad (6.23)$$

It is easily observed that the Lagrangian (6.22) is semiregular.

The Legendre morphism (6.23) is an affine morphism over  $Y$ . The corresponding linear morphism over  $Y$  is

$$\overline{L} : VY \rightarrow V^*Y, \quad p_i \circ \overline{L} = a_{ij}\dot{y}^j,$$

where  $\dot{y}^j$  are bundle coordinates of the vector bundle (2.16). In particular, if  $L$  is regular, the morphism  $\overline{L}$  defines a nondegenerate fibre metric on  $VY$ .

Let us assume that the Lagrangian is almost regular (see Proposition 6.3) and that the Lagrangian constraint space  $Q$  defined by the Legendre morphism (6.23) contains the image of the zero section  $\widehat{0}(Y)$  of the Legendre bundle  $\Pi \rightarrow Y$ . It is immediately observed that  $\text{Ker } \widehat{L} = \widehat{L}^{-1}(\widehat{0}(Y))$  is an affine subbundle of the jet bundle  $J^1Y \rightarrow Y$ .

The following two ingredients in our construction play a prominent role.

(i) There exists a connection  $\Gamma$  on  $Y \rightarrow X$  which takes its values into  $\text{Ker } \widehat{L}$ :

$$\Gamma : Y \rightarrow \text{Ker } \widehat{L}, \quad \boxed{a_{ij}\Gamma^j + b_i = 0.} \quad (6.24)$$

With this connection, the Lagrangian (6.22) can be brought into the form

$$\mathcal{L} = \frac{1}{2}a_{ij}(y_t^i - \Gamma^i)(y_t^j - \Gamma^j) + c'.$$

For instance, if it is regular, the connection (6.24) is unique.

(ii) There exists a linear morphism

$$\sigma : V^*Y \rightarrow VY, \quad \dot{y}^i \circ \sigma = \sigma^{ij}p_j \quad (6.25)$$

such that

$$\overline{L} \circ \sigma|_Q = \text{Id}_Q, \quad \boxed{a_{ij}\sigma^{jk}a_{kb} = a_{ib}.}$$

Then, the jet bundle  $J^1Y \rightarrow Y$  has the splitting

$$\boxed{J^1Y = \text{Ker } \widehat{L} \oplus_Y \text{Im } \sigma,} \quad y_t^i = [y_t^i - \sigma^{ik}(a_{kj}y_t^j + b_k)] + [\sigma^{ik}(a_{kj}y_t^j + b_k)]. \quad (6.26)$$

If the Lagrangian (6.22) is regular, the morphism (6.25) is determined uniquely.

Given the morphism  $\sigma$  (6.25) and the connection  $\Gamma$  (6.24), let us consider the Hamiltonian form

$$H = p_i dy^i - [\Gamma^i(p_i - \frac{1}{2}b_i) + \frac{1}{2}\sigma^{ij}p_i p_j - c]dt. \quad (6.27)$$

**PROPOSITION 6.6.** The Hamiltonian form (6.27) is associated with the Lagrangian (6.22). The family of these forms parameterized by the connections (6.24) constitute a complete family.  $\square$

Given the Hamiltonian (6.27), let us consider the Hamilton equations (5.10a) for sections  $r$  of the bundle  $\Pi \rightarrow X$ . They read

$$\begin{aligned} J^1 s &= (\Gamma + \sigma) \circ r, & s &= \pi_{\Pi Y} \circ r, \\ d_t r^i &= \Gamma^i + \sigma^{ij} r_j. \end{aligned} \quad (6.28)$$

With the splitting (6.26), we have the following surjections

$$\begin{aligned} \mathcal{S} &:= \text{pr}_1 : J^1 Y \rightarrow \text{Ker } \widehat{L}, & \mathcal{S} &: y_t^i \rightarrow y_t^i - \sigma^{ik}(a_{kj}y_t^j + b_k), \\ \mathcal{F} &:= \text{pr}_2 : J^1 Y \rightarrow \text{Im } \sigma, & \mathcal{F} &= \sigma \circ \widehat{L} : y_t^i \rightarrow \sigma^{ik}(a_{kj}y_t^j + b_k). \end{aligned}$$

With respect to these surjections, the Hamilton equations (6.28) break into two parts

$$\begin{aligned} \mathcal{S} \circ J^1 s &= \Gamma \circ s, & d_t r^i - \sigma^{ik}(a_{kj}d_t r^j + b_k) &= \Gamma^i, \\ \mathcal{F} \circ J^1 s &= \sigma \circ r, & \sigma^{ik}(a_{kj}d_t r^j + b_k) &= \sigma^{ik}r_k. \end{aligned} \quad (6.29)$$

The Hamilton equations (6.29) are independent of canonical momenta and play the role of constraints.

It should be noted that the Hamiltonian (6.27) differ from each other only in connections  $\Gamma$  (6.24) which lead to the different constraints (6.29).

**Remark 6.9.** We observe that a mechanical system described by a degenerate Lagrangian  $L$  appears a multi-Hamiltonian constrained system in the framework of the Hamiltonian formalism. In the spirit of the well-known Gotay algorithm in autonomous mechanics [4, 21], the Lagrangian constraint space  $Q$  can be called the primary constraint space. To properly apply this algorithm, however, one has to consider each Hamiltonian of a complete family of Hamiltonians associated with  $L$ . If  $L$  is semiregular, all these Hamiltonians coincide with each other on  $Q$ , but not the horizontal Hamiltonian vector fields (5.11). A different way is to utilize the Gotay algorithm in the framework of the Lagrangian formalism [34, 42]. One can investigate also the conditions of formal integrability [8, 31, 41] of the Hamilton equations.



Given a Hamiltonian associated with  $L$ , the corresponding Hamilton equations fail to satisfy these conditions at all points of  $Q$ . •

The relations between Lagrangian and Hamiltonian formalisms described above are broken under canonical transformations if the transition functions  $y^i \rightarrow y'^i$  depend on momenta. In the next Section, we overcome this difficulty.

## 6.5 The unified Lagrangian and Hamiltonian formalism

In case of a 1-dimensional base  $X$ , we can generalize the construction given in Remark 4.6 as follows.

Given a bundle  $Y \rightarrow X$ , let  $V^*J^1Y$  be the vertical cotangent bundle of  $J^1Y \rightarrow X$  coordinatized by  $(t, y^i, y_t^i, \dot{y}_i, \dot{y}_i^t)$  and  $J^1V^*Y$  the jet manifold of  $V^*Y \rightarrow X$  coordinatized by  $(t, y^i, p_i, y_t^i, p_{it})$ .

LEMMA 6.7. There is the isomorphism

$$\mathbf{\Pi} = V^*J^1Y = J^1V^*Y, \quad \dot{y}_i \longleftrightarrow p_{it}, \quad \dot{y}_i^t \longleftrightarrow p_i, \quad (6.30)$$

over  $J^1Y$ . □

**Proof.** The isomorphism (6.30) is proved by comparing the transition functions of the coordinates  $(\dot{y}_i, \dot{y}_i^t)$  and  $(p_i, p_{it})$ . •

Due to the isomorphism (6.30), one can think of  $\mathbf{\Pi}$  as being both the Legendre bundle (phase space) over the configuration space  $J^1Y$  and the configuration space over the phase space  $\mathbf{\Pi}$ . Hence, the space  $\mathbf{\Pi}$  can be utilized as the unified configuration and phase space of the joint Lagrangian and Hamiltonian formalism. This space is coordinatized by  $(t, y^i, y_t^i, p_{it}, p_i)$ , where  $(y^i, p_{it})$  and  $(y_t^i, p_i)$  are canonically conjugate pairs. The space  $\mathbf{\Pi}$  is equipped with the canonical form (5.4) given by the coordinate form

$$\boxed{\Lambda = (dp_{it} \wedge dy^i + dp_i \wedge dy_t^i) \wedge dt \otimes \partial_t}$$

and with the canonical form (5.20) which reads

$$\boxed{\Omega = (dp_{it} \wedge dy^i + dp_i \wedge dy_t^i) \wedge dt = d_t(dp_i \wedge dy^i \wedge dt)}. \quad (6.31)$$

As in Section 5.1, one can introduce Hamiltonian connections and Hamiltonian forms on  $\mathbf{\Pi}$ . Let

$$H = p_{it}dy^i + p_idy_t^i - \mathcal{H}(t, y^i, y_t^i, p_{it}, p_i)dt \quad (6.32)$$

be a Hamiltonian form (5.7) on  $\Pi$ . The corresponding Hamilton equations (5.10a) – (5.10b) read

$$d_t y^i = \frac{\partial \mathcal{H}}{\partial p_{it}}, \quad (6.33a)$$

$$d_t y_t^i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad (6.33b)$$

$$d_t p_i = -\frac{\partial \mathcal{H}}{\partial y_t^i}, \quad (6.33c)$$

$$d_t p_{it} = -\frac{\partial \mathcal{H}}{\partial y^i}. \quad (6.33d)$$

**Example 6.10.** Given a connection  $\Gamma$  on  $Y \rightarrow X$ , we can bring (6.32) into the form

$$H = d_t[p_i(dy^i - \Gamma^i dt)] - \widetilde{\mathcal{H}}dt = p_{it}dy^i + p_i dy_t^i - d_t(p_i \Gamma^i)dt - \widetilde{\mathcal{H}}dt,$$

where  $d_t \Gamma$  is the jet lift (6.11) of  $\Gamma$  onto  $J^1 Y \rightarrow X$ . In particular, every Hamiltonian  $\mathcal{H}_\Pi$  on  $\Pi = V^*Y$  defines the Hamiltonian

$$\mathcal{H} = d_t \mathcal{H}_\Pi = \partial_t \mathcal{H}_\Pi + y_t^i \frac{\partial \mathcal{H}_\Pi}{\partial y^i} + p_{it} \frac{\partial \mathcal{H}_\Pi}{\partial p_i} \quad (6.34)$$

on  $\Pi$  (6.30). In this case, the equations (6.33a) – (6.33d) take the form

$$d_t y^i = \frac{\partial \mathcal{H}_\Pi}{\partial p_i}, \quad (6.35a)$$

$$d_t y_t^i = d_t \frac{\partial \mathcal{H}_\Pi}{\partial p_i}, \quad (6.35b)$$

$$d_t p_i = -\frac{\partial \mathcal{H}_\Pi}{\partial y^i}, \quad (6.35c)$$

$$d_t p_{it} = -d_t \frac{\partial \mathcal{H}_\Pi}{\partial y^i}. \quad (6.35d)$$

It is easily observed that they are equivalent to the Hamilton equations (6.35a), (6.35c) for the Hamiltonian  $\mathcal{H}_\Pi$  on  $\Pi$ . ●

Substitution of (6.33a) into (6.33b) and of (6.33c) into (6.33d) leads to the equations

$$d_t \frac{\partial \mathcal{H}}{\partial p_{it}} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad (6.36a)$$

$$d_t \frac{\partial \mathcal{H}}{\partial y_t^i} = \frac{\partial \mathcal{H}}{\partial y^i} \quad (6.36b)$$

which look like the Euler–Lagrange equations for the "Lagrangian"  $\mathcal{H}$ . Though  $\mathcal{H}$  is not a true Lagrangian function, one can write  $\mathcal{H} = -\mathcal{L} + d_t(p_i\Gamma^i)$ , so that the equations (6.36a) – (6.36b) become the Euler–Lagrange equations for the Lagrangian  $\mathcal{L}$  on  $\Pi$ .

The solutions of the Hamilton equations (6.33a) – (6.33d) are obviously the solutions of the Euler–Lagrange equations (6.36a) – (6.36b), but the converse is not true.

**Example 6.11.** Let  $\mathcal{H} = -\mathcal{L}_Y + d_t(p_i\Gamma^i)$ , where  $\mathcal{L}_Y$  is a Lagrangian on  $J^1Y$ . In this case, the equations (6.36a) – (6.36b) are equivalent to the Euler–Lagrange equations (6.36b) for the Lagrangian  $\mathcal{L}_Y$ . However, their solutions fail to be solutions of the corresponding Hamilton equations (6.33a) – (6.33d) in general. ●

To give a unified picture of Examples 6.10 and 6.11, let us consider the Hamiltonian

$$\mathcal{H} = d_t\mathcal{H}_\Pi + (p_i y_t^i - \mathcal{H}_\Pi) - \mathcal{L}_Y, \quad (6.37)$$

where  $\mathcal{L}_Y$  is a semiregular Lagrangian on the configuration space  $J^1Y$  and  $\mathcal{H}_\Pi$  is a Hamiltonian associated with  $\mathcal{L}_Y$ . The corresponding Hamilton equations (6.33a) – (6.33d) read

$$d_t y^i = \frac{\partial \mathcal{H}_\Pi}{\partial p_i}, \quad (6.38a)$$

$$d_t y_t^i = d_t \frac{\partial \mathcal{H}_\Pi}{\partial p_i} + y_t^i - \frac{\partial \mathcal{H}_\Pi}{\partial p_i}, \quad (6.38b)$$

$$d_t p_i = -\frac{\partial \mathcal{H}_\Pi}{\partial y^i} - p_i + \frac{\partial \mathcal{L}}{\partial y_t^i}, \quad (6.38c)$$

$$d_t p_{it} = -d_t \frac{\partial \mathcal{H}_\Pi}{\partial y^i} + \frac{\partial \mathcal{H}_\Pi}{\partial y^i} + \frac{\partial \mathcal{L}}{\partial y^i}. \quad (6.38d)$$

Using the relations (6.17a) and (6.18), one can show that solutions of the Hamilton equations (5.10a) – (5.10b) for the Hamiltonian  $\mathcal{H}_\Pi$  which live on the Lagrangian constraint space are solutions of the equations (6.38a) – (6.38d).

In other words the equations (6.38a) – (6.38d) on the constraint subspace

$$y_t^i = \frac{\partial \mathcal{H}_\Pi}{\partial p_i}, \quad p_i = \frac{\partial \mathcal{L}}{\partial y_t^i}$$

on  $\Pi$  are equivalent to the Hamilton equations (6.35a) – (6.35d).

Now let us consider the Euler–Lagrange equations (6.36a) – (6.36b) for the Hamiltonian (6.37). They read

$$d_t y^i - \frac{\partial \mathcal{H}_\Pi}{\partial p_i} = 0, \quad (6.39a)$$

$$d_t p_i - d_t \frac{\partial \mathcal{L}_Y}{\partial y_t^i} = -\frac{\partial \mathcal{H}_\Pi}{\partial y^i} - \frac{\partial \mathcal{L}_Y}{\partial y^i}. \quad (6.39b)$$

In accordance with Proposition 4.7 and Corollary 6.4, every solution  $s$  of the Euler–Lagrange equations for the Lagrangian  $\mathcal{L}_Y$  such that the relation (4.25) holds are solutions of the equations (6.39a) – (6.39b).

In particular, if the Lagrangian  $\mathcal{L}_Y$  is hyperregular, the equations (6.38a) – (6.38d) and the equations (6.39a) – (6.39b) are equivalent to the corresponding Hamilton equations and the Euler–Lagrange equations for  $\mathcal{L}_Y$  and the associated Hamiltonian.

**Example 6.12.** Let us consider the Hamiltonian form

$$H = p_{it}(dy^i - \gamma^i dt) + p_i(dy_t^i - \gamma_t^i dt) \quad (6.40)$$

on  $\Pi$ , where  $\gamma$  is the connection (6.7) on  $J^1Y \rightarrow X$ . The associated Hamilton equations (6.33a), (6.33b) read

$$\begin{aligned} d_t y^i &= \gamma^i = y_t^i = V^* \gamma^i, & d_t y_t^i &= \gamma_t^i = A_t^i + A_{tj}^i y_t^j = V^* \gamma_t^i, \\ d_t p_i &= -p_j \frac{\partial \gamma_t^j}{\partial y_t^i} - p_{it} = V^* \gamma_i, & d_t p_{it} &= -p_j \frac{\partial \gamma_t^j}{\partial y^i} = V^* \gamma_i t, \end{aligned} \quad (6.41)$$

where  $V^* \gamma$  is the covertical connection (2.32) on  $\Pi = V^* J^1Y$ . The equations (6.41) recover the evolution equation (6.6) which consists with the Euler–Lagrange equation (6.36a). ●

Turn now to the Poisson structure generated on  $\Pi$  by the canonical form  $\Omega$  (6.31). The corresponding Poisson bracket (5.18) reads

$$\{f, g\}_V = \frac{\partial f}{\partial p_i t} \frac{\partial g}{\partial y^i} + \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial y_t^i} - \frac{\partial g}{\partial p_i t} \frac{\partial f}{\partial y^i} + \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial y_t^i}. \quad (6.42)$$

In particular, if  $f$  is a function on  $\Pi$  and  $\mathcal{H}$  is the Hamiltonian (6.34), the Hamiltonian evolution equation (5.24) consists with that for the Hamiltonian  $\mathcal{H}_\Pi$ . If  $f$  is a function on  $J^1Y$  and  $\mathcal{H}$  is the Hamiltonian (6.40), the Hamiltonian evolution equation consists with the evolution equation

$$d_{Ht} f = \partial_\gamma \rfloor df = \partial_t f + y_t^i \partial_i f + (A_t^i + A_{tj}^i y_t^j) \partial_i^t f.$$

It is readily observed that the canonical form (6.31) and the Poisson bracket (6.42) are invariant under the canonical transformations of  $\Pi = J^1\Pi$  generated by the canonical transformations of  $\Pi$ .

## 7 Conservation laws and integrals of motion

In autonomous mechanics, an integral of motion, by definition, is a functions on the phase space whose Poisson bracket with a Hamiltonian is equal to zero. We can not extend this description to time-dependent mechanics since the Hamiltonian evolution equation (5.24) is not reduced to the Poisson bracket.

To discover integrals of motion in time-dependent mechanics, we follow the field theory approach, where the first variational formula of the calculus of variations can be utilized in order to discover differential conservation laws. This formula provides the canonical decomposition of the Lie derivative of a Lagrangian along vector fields corresponding to infinitesimal gauge transformations into two terms. The first one contains the variational derivatives and vanishes on shell. The other term is the divergence of the corresponding symmetry flow  $\mathcal{T}$ . If a Lagrangian is gauge-invariant, its Lie derivative is equal to zero and the weak conservation law  $0 \approx d_\lambda \mathcal{T}^\lambda$  holds on shell.

### 7.1 Lagrangian conservation laws

In field theory, differential conservation laws are derived from the condition of Lagrangians to be invariant under 1-parameter groups of gauge transformations.

By a gauge transformation is meant an isomorphism  $\Phi$  of a bundle  $\pi : Y \rightarrow X$  over a diffeomorphism  $f$  of  $X$ . Every 1-parameter group  $\Phi[\alpha]$  of isomorphisms of  $Y \rightarrow X$  yields the complete vector field

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i \quad (7.1)$$

which is the generator of  $\Phi[\alpha]$ . It is projected onto the vector field  $\tau = u^\mu\partial_\mu$  on  $X$  which is the generator of  $f[\alpha]$ . Conversely, one can think of any projectable vector field (7.1) on a bundle  $Y$  as being the generator of a local 1-parameter gauge group. Using the canonical lift (2.19) of  $u$  onto  $J^1Y$ , we have

$$\mathbf{L}_{\bar{u}}L = d(u]L) + u]dL = [\partial_\lambda u^\lambda \mathcal{L} + (u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda) \mathcal{L}] \omega. \quad (7.2)$$

The first variational formula provides the canonical decomposition (4.1) of the Lie derivative (7.2) in accordance with the variational task. It is given by the coordinate relation

$$\begin{aligned} \partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda] \mathcal{L} \equiv \\ (u^i - y_\mu^i u^\mu)(\partial_i - d_\lambda \partial_i^\lambda) \mathcal{L} - d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}], \end{aligned} \quad (7.3)$$

where

$$\mathcal{T} = \mathcal{T}^\lambda \omega_\lambda = [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}] \omega_\lambda, \quad \pi_i^\lambda = \partial_i^\lambda \mathcal{L}, \quad \omega_\lambda = \partial_\lambda ] \omega, \quad (7.4)$$

is the symmetry flow along the vector field  $u$ .

The first variational formula (7.3) on shell (4.4) comes to the weak transformation law

$$\begin{aligned} \partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda] \mathcal{L} \\ \approx -d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}]. \end{aligned} \quad (7.5)$$

If the Lie derivative  $\mathbf{L}_{\bar{u}} L$  (7.2) vanishes, we have the conservation law

$$0 \approx d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}].$$

It is brought into the differential conservation law

$$0 \approx \frac{d}{dx^\lambda} (\pi_i^\lambda (u^\mu \partial_\mu s^i - u^i) - u^\lambda \mathcal{L})$$

on solutions  $s$  of the Euler–Lagrange equations (4.5)

Background fields break conservation laws as follows. Let us consider the product  $Y \times Y'$  of a bundle  $Y$  coordinatized by  $(x^\lambda, y^i)$  whose sections are dynamical fields on shell (4.4) and a bundle  $Y'$  coordinatized by  $(x^\lambda, y^A)$  whose sections are background fields which take the background values  $y^B = \phi^B(x)$ ,  $y_\lambda^B = \partial_\lambda \phi^B(x)$ . Let

$$u = u^\lambda(x) \partial_\lambda + u^A(x^\mu, y^B) \partial_A + u^i(x^\mu, y^B, y^j) \partial_i \quad (7.6)$$

be a projectable vector field on  $Y \times Y'$  which is projected also onto  $Y'$  (gauge transformations of background fields do not depend on the dynamic ones). Substitution of (7.6) into (7.3) leads to the first variational formula in the presence of background fields. The weak identity

$$\begin{aligned} \partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^A \partial_A + u^i \partial_i + (d_\lambda u^A - y_\mu^A \partial_\lambda u^\mu) \partial_A^\lambda + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda] \mathcal{L} \approx \\ -d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}] + (u^A - y_\lambda^A u^\lambda) \partial_A \mathcal{L} + \pi_A^\lambda d_\lambda (u^A - y_\mu^A u^\mu) \end{aligned}$$

holds on shell (4.4). If a total Lagrangian is gauge-invariant, we discover the transformation law

$$0 \approx -d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}] + (u^A - y_\lambda^A u^\lambda) \partial_A \mathcal{L} + \pi_A^\lambda d_\lambda (u^A - y_\mu^A u^\mu) \quad (7.7)$$

in the presence of background fields.

**Remark 7.1.** The transformation law (7.7) can also be applied when the dynamical equations are not Lagrangian, but are given, e.g., by local expression

$$(\partial_i - d_\lambda \partial_i^\lambda) \mathcal{L} + F_i(t, y^j, y_t^j) = 0. \quad (7.8)$$

In this case, the transformation law reads

$$F_i \approx -d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i)]. \quad (7.9)$$

●

## 7.2 Energy-momentum conservation laws

The transformation law (7.5) is linear in the vector field  $u$ . Hence, one can consider superposition of the transformation laws along different vector fields.

Every vector field  $u$  on  $Y$  projected onto a vector field  $\tau$  on  $X$  is the sum of the lift of  $\tau$  onto  $Y$  and of some vertical vector field  $\vartheta$  on  $Y$ . It follows that every transformation law (7.5) is the superposition of the Noether transformation law

$$[\vartheta^i \partial_i + d_\lambda \vartheta^i \partial_i^\lambda] \mathcal{L} \approx d_\lambda (\pi_i^\lambda \vartheta^i)$$

for the Noether flow  $\mathcal{T}^\lambda = -\pi_i^\lambda \vartheta^i$  and of the stress-energy-momentum (SEM) transformation law [19, 20, 49].

A vector field  $\tau$  on  $X$  can be lifted to  $Y$  only by means of a connection on  $Y$ .

Let  $\tau = \tau^\mu \partial_\mu$  be a vector field on  $X$  and  $\tau_\Gamma = \tau^\mu (\partial_\mu + \Gamma_\mu^i \partial_i)$  its horizontal lift onto  $Y$  by a connection  $\Gamma$ . The weak identity (7.5) along  $\tau_\Gamma$  reads

$$\begin{aligned} \partial_\mu \tau^\mu \mathcal{L} + [\tau^\mu \partial_\mu + \tau^\mu \Gamma_\mu^i \partial_i + (d_\lambda (\tau^\mu \Gamma_\mu^i) - y_\mu^i \partial_\lambda \tau^\mu) \partial_i^\lambda] \mathcal{L} \approx \\ -d_\lambda [\pi_i^\lambda \tau^\mu (y_\mu^i - \Gamma_\mu^i) - \delta_\mu^\lambda \tau^\mu \mathcal{L}], \end{aligned} \quad (7.10)$$

where

$$\mathcal{T}_\Gamma^\lambda{}_\mu = \pi_i^\lambda (y_\mu^i - \Gamma_\mu^i) - \delta_\mu^\lambda \mathcal{L}$$

is the SEM tensor relative to the connection  $\Gamma$ .

One may choose different connections  $\Gamma$  in order to discover SEM conservation laws. The SEM flows relative to  $\Gamma$  and  $\Gamma'$  differ from each other in the Noether flow along the vertical vector field  $\vartheta = \tau^\mu (\Gamma_\mu^i - \Gamma_\mu'^i) \partial_i$ .

If the transformation law (7.10) holds for any vector field  $\tau$  on  $X$ , we come to the system of weak equalities

$$(\partial_\mu + \Gamma_\mu^i \partial_i + d_\lambda \Gamma_\mu^i \partial_i^\lambda) \mathcal{L} \approx -d_\lambda \mathcal{T}_\Gamma^\lambda{}_\mu.$$

For instance, if we choose the locally trivial connection  $\Gamma_{0\mu}^i = 0$ , then the identity (7.10) recovers the well-known transformation law

$$\frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{d}{dx^\lambda} \mathcal{T}_0^\lambda{}_\mu(s) \approx 0, \quad \mathcal{T}_0^\lambda{}_\mu(s) = \pi_i^\lambda \partial_\mu s^i - \delta_\mu^\lambda \mathcal{L}, \quad (7.11)$$

of the canonical energy-momentum tensor  $\mathcal{T}_0$ . Though it is not a true tensor, the transformation law (7.11) on solutions  $s$  of differential Euler–Lagrange equations is well-defined.

### 7.3 Hamiltonian conservation laws

To discover the conservation laws in the framework of the Hamiltonian formalism, we go back to Remark 4.6 [49].

Given a Hamiltonian form  $H$  (4.19) on the Legendre bundle  $\Pi$ , let us consider the Lagrangian (4.27) on  $J^1\Pi$ . One can apply the first variational formula (7.3) to this Lagrangian in order to get the differential conservation laws in the framework of the polysymplectic Hamiltonian formalism.

Every projectable vector field  $u$  on  $Y \rightarrow X$  can be lifted to  $\Pi$  as follows:

$$\tilde{u} = u^\mu \partial_\mu + u^i \partial_i + (-\partial_i u^j p_j^\lambda - \partial_\mu u^\mu p_i^\lambda + \partial_\mu u^\lambda p_i^\mu) \partial_\lambda^\lambda. \quad (7.12)$$

In case of the vector field  $\tilde{u}$  (7.12) and the Lagrangian  $L_H$  (4.27), the first variational formula (7.3) on shell (4.22a) – (4.22b) takes the form

$$\begin{aligned} p_i^\lambda y_\lambda^i \partial_\mu u^\mu - \partial_\lambda (u^\lambda \mathcal{H}) - u^i \partial_i \mathcal{H} + (d_\lambda u^i - \partial_\mu^\lambda \mathcal{H} \partial_\lambda u^\mu) p_i^\lambda \\ \approx d_\lambda [p_i^\lambda (u^i - \partial_\mu^\lambda \mathcal{H} u^\mu) + u^\lambda (p_i^\mu \partial_\mu^\lambda \mathcal{H} - \mathcal{H})]. \end{aligned} \quad (7.13)$$

If  $\mathbf{L}_{\tilde{u}} L_H = 0$ , then we get the weak conservation law

$$0 \approx d_\lambda [p_i^\lambda (u^i - \partial_\mu^\lambda \mathcal{H} u^\mu) + u^\lambda (p_i^\mu \partial_\mu^\lambda \mathcal{H} - \mathcal{H})] \omega. \quad (7.14)$$

On solutions  $r$  of the Hamilton equations, the weak equality (7.14) comes to the weak differential conservation law

$$0 \approx -\frac{d}{dx^\lambda} \tilde{\mathcal{T}}^\lambda(r) \omega$$

of the flow

$$\tilde{\mathcal{T}}^\lambda(r) = -[r_i^\lambda (u^i - \partial_\mu^\lambda \mathcal{H} u^\mu) + u^\lambda (r_i^\mu \partial_\mu^\lambda \mathcal{H} - \mathcal{H})].$$

The following assertion describes the relations between differential conservation laws in Lagrangian and Hamiltonian formalisms.

**PROPOSITION 7.1.** Let a Hamiltonian form  $H$  be associated with a semiregular Lagrangian  $L$ . Let  $r$  be a solution of the Hamilton equations for  $H$  which lives on the Lagrangian constraint space  $Q$  and  $s$  the associated solution of the Euler–Lagrange equations for  $L$  so that they satisfy the conditions (4.26). In virtue of the relations (4.23b) and (4.24), we have

$$\tilde{\mathcal{T}}(r) = \mathcal{T}(\widehat{H} \circ r), \quad \tilde{\mathcal{T}}(\widehat{L} \circ J^1 s) = \mathcal{T}(s), \quad (7.15)$$

where  $\mathcal{T}$  is the flow (7.4).  $\square$



In particular, let  $\tau$  be a vector field on  $X$  and  $\tau_\Gamma$  its horizontal lift onto  $Y \rightarrow X$  by a connection  $\Gamma$  on  $Y$ . We have the corresponding flow

$$\tilde{\mathcal{T}}_\Gamma^\lambda{}_\mu = p_i^\lambda \partial_\mu^i \tilde{\mathcal{H}}_\Gamma - \delta_\mu^\lambda (p_i^\nu \partial_\nu^i \tilde{\mathcal{H}}_\Gamma - \tilde{\mathcal{H}}_\Gamma), \quad (7.16)$$

where  $\tilde{\mathcal{H}}_\Gamma$  is the Hamiltonian in the splitting (4.19) of  $H$  with respect to the connection  $\Gamma$ . The relations (7.15) shows that, on the Lagrangian constraint space  $Q$ , the flow (7.16) can be treated as the Hamiltonian SEM flow relative the connection  $\Gamma$ .

The weak transformation law (7.13) of the Hamiltonian SEM flow (7.16) takes the form

$$-(\partial_\mu + \Gamma_\mu^j \partial_j - p_i^\lambda \partial_j \Gamma_\mu^i \partial_\lambda^j) \tilde{\mathcal{H}}_\Gamma + p_i^\lambda R_{\lambda\mu}^i \approx -d_\lambda \tilde{\mathcal{T}}_\Gamma^\lambda{}_\mu.$$

Let us now consider the transformation law (7.13) when the vector field  $\tilde{u}$  on  $\Pi$  is the horizontal lift of a vector field  $\tau$  on  $X$  by means of Hamiltonian connection on  $\Pi \rightarrow X$  which is associated with the Hamiltonian form  $H$ . We have

$$\tilde{u} = \tau^\mu (\partial_\mu + \partial_\mu^i \mathcal{H} \partial_i + \gamma_{i\mu}^\lambda \partial_\lambda^i).$$

In this case, the corresponding SEM flow reads

$$\tilde{\mathcal{T}}^\lambda = -\tau^\lambda (p_i^\mu \partial_\mu^i \mathcal{H} - \mathcal{H}),$$

and the weak transformation law takes the form

$$-\partial_\mu \mathcal{H} + d_\lambda (\partial_\mu^i \mathcal{H} p_i^\lambda) \approx \partial_\mu (p_i^\lambda \partial_\lambda^i \mathcal{H} - \mathcal{H}). \quad (7.17)$$

A glance at the expression (7.17) shows that the SEM flow is not conserved, but we can write the transformation law

$$-\partial_\mu \mathcal{H} + d_\lambda [\partial_\mu^i \mathcal{H} p_i^\lambda - \delta_\mu^\lambda (p_i^\nu \partial_\nu^i \mathcal{H} - \mathcal{H})] \approx 0.$$

This is exactly the Hamiltonian form of the canonical energy-momentum transformation law (7.11) in the Lagrangian formalism.

## 7.4 Integrals of motion in time-dependent mechanics

In Lagrangian mechanics when  $X$  is a 1-dimensional manifold, we consider conservation law along a vector field

$$u = u^t \partial_t + u^i \partial_i, \quad u^t = 0, 1, \quad (7.18)$$

on  $Y \rightarrow X$ . Its jet lift (2.19) onto  $J^1 Y$  reads

$$\bar{u} = u^t \partial_t + u^i \partial_i + d_t u^i \partial_i^t.$$

In this case, the first variational formula (7.3) takes the form

$$\bar{u}]d\mathcal{L} \equiv (u^i - u^t y_t^i)(\partial_i - d_t \partial_i^t) \mathcal{L} - d_t \mathcal{T}, \quad (7.19)$$

where

$$\mathcal{T} = \pi_i(u^t y_t^i - u^i) - u^t \mathcal{L} \quad (7.20)$$

is the flow along the vector field  $u$ .

The first variational formula (7.19) on shell (4.4) comes to the weak transformation law

$$\bar{u}]d\mathcal{L} \approx -d_t \mathcal{T}. \quad (7.21)$$

If the Lie derivative

$$\mathbf{L}_{\bar{u}} L = (ol u]d\mathcal{L})dt = (u^t \partial_t + u^i \partial_i + d_t u^i \partial_i^t) \mathcal{L} dt$$

vanishes, we have the conservation law

$$0 \approx d_t [\pi_i(u^t y_t^i - u^i) - u^t \mathcal{L}].$$

It is brought into the differential conservation law

$$0 \approx \frac{d}{dt} (\pi_i(u^t \partial_t s^i - u^i) - u^t \mathcal{L})$$

on solutions  $s$  of the Euler–Lagrange equations. A glance at this expression shows that, in mechanics, the conserved flow (7.20) plays the role of a (first) integral of motion.

Every transformation law (7.21) along a vector field  $u$  (7.18) on  $Y$  can be represented as superposition of the Noether transformation law along a vertical vector field  $u$ , where  $u^t = 0$  and of the energy transformation law along a horizontal lift

$$\tau_\Gamma = \partial_t + \Gamma^i \partial_i \quad (7.22)$$

of the standard vector field  $\partial_t$  on  $X$  by means some connection  $\Gamma$  on  $Y \rightarrow X$  [19, 15].

If  $u$  is a vertical vector field, the transformation law (7.21) reads

$$(u^i \partial_i + d_t u^i \partial_i^t) \mathcal{L} \approx d_t (\pi_i u^i).$$

If the Lie derivative of  $L$  along  $u$  is equal to zero, we have the integral of motion  $\mathcal{T} = \pi_i u^i$ .

**Example 7.2.** Let a Lagrangian  $\mathcal{L}$  does not depend on some coordinate  $y^1$ . Then, its Lie derivative along the vertical vector field  $u = \partial_1$  is equal to zero, and the corresponding integral of motion is the momentum  $\mathcal{T} = \partial_1^t \mathcal{L}$ . ●

The transformation law (7.21) along the horizontal lift  $\tau_\Gamma$  (7.22) takes the form

$$(\partial_t + \Gamma^i \partial_i + d_t \Gamma^i \partial_i^t) \mathcal{L} = -d_t(\pi_i(y_t^i - \Gamma^i) - \mathcal{L}), \quad (7.23)$$

where

$$E_\Gamma = \pi_i(y_t^i - \Gamma^i) - \mathcal{L} \quad (7.24)$$

is the energy density. Obviously, it depends on the choice of the connection  $\Gamma$ . From the physical viewpoint, one can treat  $\Gamma$  as a (local) reference frame,  $\dot{y}_\Gamma^i = y_t^i - \Gamma^i$  and  $E_\Gamma$  as the relative velocities and the energy density respectively with regard to this reference frame.

**Example 7.3.** Let us put  $\Gamma = 0$  that corresponds to choice of the (local) reference frame given by the coordinates  $y^i$ . In this case, the energy transformation law (7.23) takes the familiar form

$$\partial_t \mathcal{L} = -d_t(\pi_i y_t^i - \mathcal{L}), \quad (7.25)$$

where  $y_t^i$  can be treated as velocities with respect to the above-mentioned reference frame. ●

**Example 7.4.** Let us consider the bundle  $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  coordinatized by  $(t, y)$ . It describes 1-dimensional motion.

$$\Gamma^i = a^i t \quad (7.26)$$

be a connection on this bundle which defines an accelerated reference frame with respect to the reference frame  $y$ . Consider the Lagrangian

$$L = \frac{1}{2}(y_t^i - a^i t)^2 dt$$

which describes the free particle relative to the reference frame  $\Gamma$ . It is easy to see that the energy density (7.24) relative to connection (7.26) is conserved. It is exactly the energy of the free particle with respect to the reference frame  $\Gamma$ . ●

We now turn to the Hamiltonian mechanics.

Given a vector field (7.18), let

$$\tilde{u} = u^t \partial_\mu + u^i \partial_i - \partial_i u^j p_j \partial^i, \quad u^t = 0, 1, \quad (7.27)$$

be its lift onto the phase space  $V^*Y$ . We consider conservation laws in time-dependent Hamiltonian mechanics along the vector fields (7.27). As a particular case of the transformation law (7.13), we have

$$-u^t \partial_t \mathcal{H} - u^i \partial_i \mathcal{H} + p_i d_t u^i \approx d_t(p_i u^i - u^t \mathcal{H}). \quad (7.28)$$

In case of a vertical vector field  $u$ , this transformation law comes to the weak equality

$$-u^i \partial_i \mathcal{H} \approx u^i d_t p_i.$$

In particular, if a Hamiltonian  $\mathcal{H}$  is locally independent on the coordinate  $y^i$ , the momentum  $p_i$  is the (local) integral of motion.

The transformation law (7.28) along the horizontal lift  $\tau_\Gamma$  (7.22) takes the form

$$-\partial_t \mathcal{H} - \Gamma^i \partial_i \mathcal{H} + p_i d_t \Gamma^i \approx -d_t \widetilde{\mathcal{H}}_\Gamma.$$

It follows that, in accordance with Proposition 7.1, the Hamiltonian partner of the Lagrangian energy density  $E_\Gamma$  (7.24) is the Hamiltonian function  $\widetilde{\mathcal{H}}_\Gamma$  from the splitting (5.7). Therefore, we can treat it as the energy function with respect to the reference frame  $\Gamma$ . In particular, if  $\Gamma^i = 0$ , we get the well-known energy transformation law

$$\partial_t \mathcal{H} = d_t \mathcal{H}$$

which is the Hamiltonian variant of the Lagrangian law (7.25).

## 8 References

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